IRE Transactions on INFORMATION THEORY

A Journal Devoted to the Theoretical and Experimental Aspects of Information Transmission, Processing and Utilization.

Volume IT-7 OCTOBER, 1961 Number 4

In This Issue

PUBLISHED BY THE
Professional Group on Information Theory

IRE Professional Group on Information Theory

The Professional Group on Information Theory is an organization, within the framework of the IRE, of members with principal professional interest in Information Theory. All members of the IRE are eligible for membership in the Group and will receive all Group publications upon payment of an annual fee of $4.00.

IRE TRANSACTIONS® on INFORMATION THEORY is published in January, April, July, and October, by the IRE for the Professional Group on Information Theory, at 1 East 79 Street, New York 21, N. Y. In addition to these regular quarterly issues, Special Issues appear from time to time. Responsibility for contents rests upon the authors and not upon the IRE, the Group, or its members. Individual copies of this issue and all available back issues, except PGIT-4, may be purchased at the following prices: IRE members (one copy) $2.25, libraries and colleges $3.25, all others $4.50. Annual subscription rate: non-members $17.00; libraries and colleges, $12.75.

INFORMATION THEORY

IRE Transactions on Information Theory

A Journal Devoted to the Theoretical and Experimental Aspects of Information Transmission, Processing and Utilization

Volume IT-7 | October, 1961 | Number 4

Published Quarterly

TABLE OF CONTENTS

Optimum Linear Finite-Dimensional Estimator of Signal Waveforms*

TADAO KASAMI†

Summary—This paper deals with the linear estimation of noise corrupted signal waveforms, from observation over a specified finite time-interval. Two new features are delineated in this paper: 1) the estimating operator is assumed to be finite-dimensional, because economically realizable operators are finite-dimensional in general, and 2) the cost of observation and estimation is taken into account, and assumed to be dependent upon the dimension of the operator only. As a result, the optimum linear finite-dimensional operator that minimizes the risk (the sum of cost and average loss) is obtained for the case in which a quadratic loss function due to error is adopted.

I. Introduction

THIS PAPER is concerned with the following problem. A signal process, $X(t)$, is corrupted by noise. $Y(t)$, the signal corrupted by noise, is observed continuously over a specified finite time-interval I_2. Signal and noise are nonstationary in general. Then the true signal waveform on a time-interval I_1 is to be estimated on the basis of the observed values of $Y(t)$ over I_2.

Many authors have developed the nonstationary smoothing and prediction theory [2]–[10]. The present approach is related to that of Middleton [9], David [4], Bendat [8], and Kramer-Mathews [7], among others.

Where finite data are used, the operator selected should (based upon an appropriate criterion) be the one providing the best *estimated* signal waveform corresponding to the *observed* waveform. Middleton considered this problem from the viewpoint of decision theory and assumed that infinite numbers of linear time-invariant filters can be employed [9]. In most practical situations, however, the operations that can be realized physically and economically are finite-dimensional. By a finite-dimensional operator,[1] we mean an operator whose range is a finite-dimensional space. (See Section III.) Also, the cost of observation and estimation must be taken into account. In Section III a simple cost function will be introduced.

In most situations there is little reliable information about the statistics of the waveforms of signal and noise except, perhaps, for the second-order properties, so that elaborate nonlinear estimates, requiring additional statistical description of the signal and noise, cannot be applied. For this reason, and for the sake of simplicity, a quadratic loss function due to error is utilized, and the discussion will be restricted to linear estimates. In Sections IV and V, the optimum linear finite-dimensional estimate that minimizes the risk (the sum of cost and average loss) will be derived.

II. Representation of the Original Signal Process and the Observed Signal Process

Let $X(t)$ denote the original signal process on an interval $I_1 = [0, T] (0 < T < \infty)$, and let $Y(t)$ denote the observed signal process (received signal corrupted by noise[2]) on an interval $I_2 = [T_1, T_2] (-\infty < T_1 < T_2 < \infty)$.

Without loss of generality, we may assume that

$$E\,X(t) = 0, \qquad E\,Y(t) = 0, \tag{1}$$

where E denotes the mathematical mean. Also, it is assumed that:

$$E\,X^2(t) < \infty, \qquad (t \in I_1), \tag{2}$$

$X(t)$ is continuous in quadratic mean,

$Y(t)$ can be represented as follows,

$$Y(t) = Y_1(t) + Y_2(t), \tag{3}$$

$$E\,Y_1(t) = 0, \qquad E\,Y_1^2(t) < \infty, \tag{4}$$

where $Y_1(t)$ is continuous in quadratic mean, and $Y_2(t)$ is either white Gaussian noise statistically independent of $X(t)$ and $Y_1(t)$, or vanishes identically; and $K_s(t_1, t_2)$, $K_i(t_1, t_2)$, $K_{i_1}(t_1, t_2)$, and $K_M(t_1, t_2)$ are known and defined by

$$\begin{aligned} K_s(t_1, t_2) &= EX(t_1)X(t_2), && (t_1, t_2 \in I_1), \\ K_i(t_1, t_2) &= EY(t_1)Y(t_2), && (t_1, t_2 \in I_2), \\ K_{i_1}(t_1, t_2) &= EY_1(t_1)Y_1(t_2), && (t_1, t_2 \in I_2), \end{aligned} \tag{5}$$

$$K_M(t_1, t_2) = EX(t_1)Y(t_2) = E\,X(t_1)Y_1(t_2) \quad (t_1 \in I_1, t_2 \in I_2).$$

Since K_s and K_{i_1} are continuous non-negative-definite functions, we shall use Mercer's theorem [12]. Then

$$K_s(t_1, t_2) = \sum_m \mu_m \phi_m(t_1)\phi_m(t_2); \qquad \mu_m \geq \mu_{m+1} > 0, \tag{6}$$

$$K_{i_1}(t_1, t_2) = \sum_m \sigma_m^{(1)} \psi_m^{(1)}(t_1)\psi_m^{(1)}(t_2); \quad \sigma_m^{(1)} \geq \sigma_{m+1}^{(1)} > 0, \tag{7}$$

where the series converge absolutely and uniformly on $I_1 \times I_1$ and $I_2 \times I_2$, respectively, and the continuous

* Received by the PGIT, May 6, 1960; revised manuscript received, October 14, 1960. The present paper is based on an earlier work of the author [1].

† Faculty of Engrg., Osaka University, Osaka, Japan.

[1] A finite-dimensional operator is also called a degenerate operator or a dyad.

[2] $Y(t)$ may or may not be the signal plus noise.

functions $\phi_m[\psi_m^{(1)}]$ are eigenfunctions of $K_s(K_{i_1})$ corresponding to eigenvalues $\mu_m[\sigma_m^{(1)}]$:

$$\int_0^T K_s(t_1, t_2)\phi_m(t_2)\, dt_2 = \mu_m\phi_m(t_1), \tag{8}$$

$$\int_{T_1}^{T_2} K_{i_1}(t_1, t_2)\psi_m^{(1)}(t_2)\, dt_2 = \sigma_m^{(1)}\psi_m^{(1)}(t_1). \tag{9}$$

Eigenfunctions which correspond to (necessarily finitely) multiple eigenvalues are written with distinct indexes, and all eigenfunctions are orthonormalized:

$$\int_0^T \phi_m(t)\phi_n(t)\, dt = \delta_{mn}, \quad \int_{T_1}^{T_2} \psi_m^{(1)}(t)\psi_n^{(1)}(t)\, dt = \delta_{mn}. \tag{10}$$[3]

Then $X(t)$ and $Y_1(t)$ have the following orthogonal decompositions (see [12]):

$$X(t) = \sum_n x_n\phi_n(t), \tag{11}$$

$$Y_1(t) = \sum_n y_n^{(1)}\psi_n^{(1)}(t) \tag{12}$$

with

$$Ex_mx_n = \mu_n\,\delta_{mn}, \qquad Ey_m^{(1)}y_n^{(1)} = \sigma_n^{(1)}\,\delta_{mn}. \tag{13}$$

Let us define H_s as the linear subspace of $L_2(I_1)$ spanned by all ϕ_n's in (6). Note that

$$\int_0^T K_s(t, t)\, dt = \sum_n \mu_n = \sum_n Ex_n^2 < \infty.$$

Then we see that a sample of $X = (x_1, x_2, \cdots)$ belongs to H_s with probability 1. For the case in which $Y_2 \equiv 0$ in (3), let $\psi_n = \psi_n^{(1)}$ and $y_n = y_n^{(1)}$, and let H_i denote the linear subspace of $L_2(I_2)$ spanned by all ψ_n's. Then a sample of $Y = (y_1, y_2, \cdots)$ belongs to H_i with probability 1. Also in case $Y_2 \not\equiv 0$, let $\{\psi_n(t)\}$ denote a complete orthonormal set of functions of $L_2(I_2)$ which involves all $\psi_n^{(1)}(t)$'s. We now put

$$\begin{aligned} y_n &= y_n^{(1)} + y_n^{(2)}, \\ y_n^{(1)} &= \int_{T_1}^{T_2} Y_1(t)\psi_n(t)\, dt, \\ y_n^{(2)} &= \sigma^{1/2}\int_{T_1}^{T_2} \psi_n(t)\, dx(t, \alpha), \qquad (\sigma > 0). \end{aligned} \tag{14}$$

Here $x(t, \alpha)$ is Wiener's Brownian Motion function [13], and

$$Ey_m^{(2)}y_n^{(2)} = \sigma\,\delta_{mn}. \tag{15}$$

Then $Y(t)$ can be represented as a random vector $Y = (y_1, y_2, \cdots)$. Let H_i denote the space consisting of all sample vectors of the random vector Y. (If $Y_2(t) \not\equiv 0$, H_i is not a Hilbert space.) Now we shall define matrices U, V, and W as

$U = (\mu_m \cdot \delta_{mn})$;

$1 \leq m, n \leq \dim H_s (= \text{the dimension of } H_s)$,

$V = (\sigma_m \cdot \delta_{mn})$;

$1 \leq m, n \leq \dim H_i (= \text{the dimension of } H_i)$,

$W = (\rho_{mn})$; $\quad 1 \leq m \leq \dim H_s, \quad 1 \leq n \leq \dim H_i$,

where

$$\begin{aligned} \sigma_m &= Ey_m^2 = E\{y_m^{(1)}\}^2 + E\{y_m^{(2)}\}^2 \\ &= E\{y_m^{(1)}\}^2 + \sigma \geq \sigma, \\ \rho_{mn} &= Ex_my_n = Ex_my_n^{(1)}. \end{aligned} \tag{16}$$

Then U and V are positive-definite type (in the strict sense) diagonal matrices.

III. Linear Finite-Dimensional Estimator

In Section II it was shown that $X(t)[Y(t)]$ can be represented as a random vector

$$X = (x_1, x_2, \cdots)[Y = (y_1, y_2, \cdots)]$$

and that samples of $X(Y)$ belong to $H_s(H_i)$ with probability 1. Therefore, a waveform estimator can be regarded as a mapping from H_i into H_s. In general, a linear M-dimensional estimate of $X(t)$, $\tilde{X}(t)$, can be expressed in the following form:

$$\tilde{X}(t) = \sum_{k=1}^{M} \Gamma_k(Y)\Phi_k(t); \qquad \Phi_k(t) \in H_s, \tag{17}$$

where $\Gamma_k(Y)$ is a linear functional defined in the entire space of H_i. It may be justified to assume that:

$$E\Gamma_k^2 < \infty, \qquad (k = 1, 2, \cdots, M), \tag{18}$$

and $\Gamma_k(Y)$ can be represented as

$$\Gamma_k(Y) = \sum_n \gamma_{kn}y_n. \tag{19}$$[4]

From (15), (18), and (19), it follows that

$$\sum_n \gamma_{kn}^2\sigma_n < \infty. \tag{20}$$

As stated in Section V, Γ_k satisfying conditions in (18) and (19) can be realized within any given quadratic mean error by the output at T_2 of a linear time-invariant filter. Thus, the practical waveform estimator defined by (17) consists of a parallel bank of M linear realizable filters, the outputs Γ_k's of which are multiplied by $\Phi_k(t_\lambda)$ and added to give the estimate of the wave at t_λ. By allowing t_λ to vary in $[0, T]$, the entire waveform may be reproduced (in estimate).

Now let H_f denote the real vector space which is of the same dimensions as H_i is, and which consists of all

[3] δ_{mn} is the Kronecker δ.

[4] $\sum_n$ converges in quadratic mean.

real sequences $z = (z_1, z_2, \cdots)$ such that

$$\sum_n \sigma_n z_n^2 < \infty . \tag{21}$$

Further, let us define the inner product of z with $z'[z' = (z_1', z_2', \cdots)]$ by

$$(z, z') = \sum_n \sigma_n z_n z_n'. \tag{22}$$

Then H_f can be regarded as a separable Hilbert space. If we put

$$e_n = (0, 0, \cdots, \overset{(n)}{1}, 0 \cdots) \tag{23}$$

then we get

$$z = \sum_n z_n e_n. \tag{24}$$

Let γ_k be defined by

$$\gamma_k = \sum_n \gamma_{kn} e_n; \tag{25}$$

then (20) implies

$$\gamma_k \, \varepsilon \, H_f, \qquad (k = 1, 2, \cdots, M). \tag{26}$$

To simplify the expression, we shall use the notation

$$[\gamma_k, Y] = \sum_n \gamma_{kn} y_n. \tag{27}$$

By using these notations, (17) can be written as

$$\tilde{X} = \sum_{k=1}^{M} [\gamma_k, Y]\Phi_k, \tag{28}$$[5]

$$\Phi_k \, \varepsilon \, H_s, \qquad \gamma_k \, \varepsilon \, H_f.$$

By a linear finite-dimensional estimate, we shall mean an estimate that can be expressed as (28).

It is clear that the cost of observation and estimation depends principally on M, the dimension of the estimator. We may reasonably assume that the cost function c depends only on M and satisfies the following relations:

$$c(M) \leq c(M + 1),$$

$$c(0) = 0; \qquad c(\infty) = \infty, \tag{29}$$

and that the loss function due to error, L, is evaluated by

$$L = l \int_0^T | X(t) - \tilde{X}(t) |^2 \, dt = l \, || X - \tilde{X} ||^2, \tag{30}$$

where l is a positive constant and $|| V ||$ denotes the norm of a vector $V \, \varepsilon \, H_s$. Let us define the risk r by

$$r = c(M) + EL. \tag{31}$$

In Section IV, the optimum estimate that minimizes the risk is derived. (In what follows, it is assumed that if $\tilde{X}_1$ and $\tilde{X}_2$ give the identical risk and $\tilde{X}_1$ gives less average loss than $\tilde{X}_2$, then $\tilde{X}_1$ is preferable.)

[5] $\tilde{X}$, ϕ_k, ψ_k and Φ_k denote, respectively, the vectors corresponding to $\tilde{X}(t)$, $\phi_k(t)$, $\psi_k(t)$, and $\Phi_k(t)$.

Note

1) "Finite-dimensional operator" in this paper corresponds to what is called dyad or degenerate operator in linear space theory [18]. In (28), it is essential that the finite-dimensional operator can be represented only by a finite number of vectors Φ_k's and γ_k's. Generally speaking, $\Phi_k(\gamma_k)$ is represented as an infinite linear sum of basis $\{\phi_n\}(\{\rho_n\})$. However, this is not essential. The bases $\{\phi_n\}$ and $\{\rho_n\}$ have been adopted simply for facilitating mathematical handling. In the optimum operator, as will be stated in Section V, $\Phi_k(t)$'s are identical with eigenfunctions $\xi_k(t)$'s as defined by (54); $w_k(t)$'s which are γ_k's expressed in terms of time functions are equal to eigenfunctions $w_{k0}(t)$'s as formally defined by (61).

2) In this paper, the "cost" does not include the expenses for calculation connected with system design. The "cost" comprises the expenses of real-time calculation, or filters, adders, function-generators, etc., necessary for realizing the operator obtained upon calculation. The problem of the expenses for calculation is given no further treatment here.

3) The assumption in (29) is an approximation introduced to simplify the theoretical approach, but it is probably an appropriate approximation. (Refer to Section VI.)

4) Where M is restricted within $1 \leq M \leq M_0$, we may put $c(M) = \infty$, $M > M_0$.

5) When $c(M)$ is not given explicitly, we may regard M as a parameter and use the results stated in the proof of the theorem (see Appendix) to know how the loss due to error decreases with increasing M by ΔM.

6) The present problem stated in terms of vector spaces may have a wider meaning. As an example, let us consider the linear estimation of a vector $X = (x_1, \cdots, x_N)$ based on an observed vector $Y = (y_1, \cdots, y_{N'})$. Suppose that N and N' are large and the components of $Y(X)$ are highly correlated. Then, if the estimate is to be obtained within a given time, it is more reasonable to arrange the data Y in appropriate forms $\Gamma_k = \sum_n \gamma_{kn} y_n (k = 1, 2, \cdots, M; M < N)$ and to estimate X from Γ_k's than to estimate X directly from Y.

IV. Optimum Linear Finite-Dimensional Estimator I

Let us define an orthonormal basis of H_f, $\{e_n'\}$, as follows:

$$e_n' = \sigma_n^{-1/2} e_n, \qquad (n = 1, 2, \cdots). \tag{32}$$[6]

If we put

$$\gamma_{kn}' = \sigma_n^{1/2} \gamma_{kn}, \tag{33}$$

then

$$\gamma_k = \sum_n \gamma_{kn} \cdot e_n = \sum_n \gamma_{kn}' \cdot e_n'. \tag{34}$$

[6] Note that $\sigma_n > 0$.

There is no loss of generality in making the assumption:

$$(\gamma_i, \gamma_k) = \delta_{ik}, \qquad (i, k = 1, 2, \cdots, M) \tag{35}$$

Since a linear transformation of γ_k's in (28) induces only a linear transformation of Φ_k's. From (19), (22) and (35), it follows that

$$E\Gamma_i\Gamma_k = \delta_{ik}, \qquad (i, k = 1, 2, \cdots, M). \tag{36}$$

We shall use the basis $\{e'_n\}$ for H_f and the basis $\{\phi_n\}$ for H_s throughout this section. Then $[\gamma_k, Y]$ becomes

$$\sum_n \sigma_n^{-1/2}\gamma'_{kn}y_n.$$

Lemma 1:[7] For a given positive integer $M(\leq \dim H_i)$ and given $\gamma_k(k = 1, 2, \cdots, M)$ satisfying (35), Φ_k's which minimize EL of the linear finite-dimensional estimate $\tilde{X}$ are given by

$$\Phi_k = WV^{-1/2}\gamma_k \qquad (k = 1, 2, \cdots, M), \tag{37}$$

where $V^{-1/2}$ is the diagonal matrix;

$$V^{-1/2} = \text{diag}\,(\sigma_1^{-1/2}, \sigma_2^{-1/2} \cdots).$$

Here,

$$\begin{aligned} EL &= l\sum_n \mu_n - l\sum_{k=1}^{M} \| \Phi_k \|^2 \\ &= l\sum_n \mu_n - l\sum_{k=1}^{M} \gamma_k^T V^{-1/2}W^TWV^{-1/2}\gamma_k. \end{aligned} \tag{38}$$

(γ_k^T and W^T denote, respectively, the transposed vector of γ_k[8] and the transposed matrix of W.)

Let $\gamma_k = e'_k$ in (37); then there results

$$\begin{aligned} \| \Phi_k \|^2 &= \| WV^{-1/2}e'_k \|^2 \\ &= \sum_n \rho_{nk}^2\sigma_k^{-1}. \end{aligned}$$

Since $L \geq 0$, it follows from (38) that

$$\sum_n \mu_n \geq \sum_{k=1}^{M}\sum_n \rho_{nk}^2 \cdot \sigma_k^{-1}.$$

Note that M in the above expression is arbitrary. Then we find

$$\sum_{n=1}^{\dim H_s}\sum_{k=1}^{\dim H_f} \rho_{nk}^2\sigma_k^{-1} \leq \sum_n \mu_n < \infty. \tag{39}$$

Now, let L_f denote the linear operator from H_f into itself which is represented by the matrix $V^{-1/2}W^TWV^{-1/2}$, and let L_s denote the linear operator from H_s into itself which is represented by the matrix $WV^{-1}W^T$. Clearly, L_f and L_s are symmetric and non-negative-definite types. Also let us define the norm $\| A \|$ of a matrix $A = (a_{ik})$ by

$$\| A \|^2 = \sum_{i,k} | a_{ik} |^2.$$

By using the Cauchy inequality and inequality (39), we obtain:

$$\begin{aligned} \| V^{-1/2}W^TWV^{-1/2} \|^2 &= \sum_p\sum_q (\sum_m \sigma_p^{-1/2}\rho_{mp}\rho_{mq}\sigma_q^{-1/2})^2 \\ &\leq \sum_p\sum_q (\sum_m \rho_{mp}^2\sigma_p^{-1})(\sum_m \rho_{mq}^2\sigma_q^{-1}) \\ &= (\sum_p\sum_m \rho_{mp}^2\sigma_p^{-1})^2 < \infty, \end{aligned} \tag{40}$$

$$\begin{aligned} \| WV^{-1}W^T \|^2 &= \sum_p\sum_q (\sum_m \rho_{pm}\sigma_m^{-1}\rho_{qm})^2 \\ &\leq \sum_p\sum_q (\sum_m \rho_{pm}^2\sigma_m^{-1})(\sum_m \rho_{qm}^2\sigma_m^{-1}) \\ &= (\sum_p\sum_m \rho_{pm}^2\sigma_m^{-1})^2 < \infty. \end{aligned} \tag{41}$$

Hence L_f and L_s are symmetric non-negative-definite-type linear operators of the Hilbert-Schmidt type [16]. Consequently, L_f and L_s have pure discrete non-negative eigenvalues [14], [16] and the nonzero eigenvalues can be arranged in order of decreasing magnitude:[9]

$$\lambda_1 \geq \lambda_2 \geq \cdots, > 0,$$

$$\lambda'_1 \geq \lambda'_2 \geq \cdots, > 0.$$

Let $\zeta_k(\xi_k)$ denote the orthonormal eigenvector of $L_f(L_s)$ corresponding to $\lambda_k(\lambda'_k)$. That is,

$$V^{-1/2}W^TWV^{-1/2}\zeta_k = \lambda_k\zeta_k, \tag{42}$$

$$(\zeta_i, \zeta_k) = \delta_{ik}, \tag{43}$$

$$WV^{-1}W^T\xi_k = \lambda'_k\xi_k, \tag{44}$$

$$(\xi_i, \xi_k) = \delta_{ik}. \tag{45}$$

Then the following lemma is verified.

Lemma 2: For any index k,

$$\lambda_k = \lambda'_k. \tag{46}$$

And further, for nonzero eigenvalues λ_k's, let $\bar{\xi}_k(\bar{\zeta}_k)$ be defined by

$$\bar{\xi}_k = \lambda_k^{-1/2}WV^{-1/2}\zeta_k, \tag{47}$$

$$(\bar{\zeta}_m = \lambda_k^{-1/2}V^{-1/2}W^T\xi_k), \tag{48}$$

then $\bar{\xi}_k$'s ($\bar{\zeta}_k$'s) are orthonormal eigenvectors of $L_s(L_f)$ corresponding to λ_k's. The following theorem can be derived from Lemmas 1 and 2.

Theorem: The optimum linear finite-dimensional estimate $\tilde{X}_{\text{opt}}$ that minimizes the risk is given by

$$\tilde{X}_{\text{opt}} = \sum_{k=1}^{M_{\text{opt}}} \lambda_k^{1/2}[\bar{\zeta}_k, Y]\xi_k; \qquad \left(\sum_{k=1}^{M_{\text{opt}}} \lambda_k^{1/2}[\zeta_k, Y]\bar{\xi}_k\right), \tag{49}$$

and the minimum risk $r_{\min}$ is given by

$$r_{\min} = c(M_{\text{opt}}) + lE \| X \|^2 - l\sum_{k=1}^{M_{\text{opt}}} \lambda_k. \tag{50}$$

[7] For the proofs of lemmas and theorem, see the Appendix.

[8] In Sections IV–VI, γ_k, e_k, $e_{k'}$, Φ_k, etc., denote column vectors.

[9] Throughout this paper multiple eigenvalues are written with distinct indexes.

Here, $\xi_k(\zeta_k)$ is defined by (44) and (45) [(42) and (43)], $\bar{\zeta}_k(\bar{\xi}_k)$ is defined by (48) [(47)], and M_{opt} is the largest of integers that minimize

$$c(M) - l \sum_{k=1}^{M} \lambda_k, \qquad (0 \leq M \leq \dim H_s).$$

This theorem shows that our present problem to find the optimum linear finite-dimensional estimate is reduced to an eigenvalue problem.

It is interesting to note that if H_s is of finite dimensions, say N, then

$$M_{\text{opt}} \leq M_0 \leq N (M_0 \text{ being the rank of } WV^{-1}W^T),$$

and if

$$c(M+1) - c(M) \geq c(M) - c(M-1) (M = 1, 2 \cdots),$$

M_{opt} is the largest of integers satisfying

$$\lambda_M \geq \frac{c(M) - c(M-1)}{l}. \tag{51}$$

V. Optimum Linear Finite-Dimensional Estimator II

Here, the results obtained in Section IV will be written in terms of time functions. $K_s(t_1, t_2)$, $K_i(t_1, t_2)$ and $K_m(t_1, t_2)$ correspond to U, V, and W, respectively. Let $F(t_1, t_2)$, $D(t_1, t_2)$ and $\xi_k(t)$ denote the functions corresponding to WV^{-1}, $WV^{-1}W^T$ and ξ_k, respectively. Then the following expressions are easily obtained:

$$\int_{T_1}^{T_2} F(t_1, t) K_i(t, t_2)\, dt = K_M(t_1, t_2). \tag{52}$$

$$D(t_1, t_2) = \int_{T_1}^{T_2} F(t_1, t) K_M(t_2, t)\, dt. \tag{53}$$

$$\int_0^T D(t, \tau) \xi_k(\tau)\, d\tau = \lambda_k \xi_k(t). \tag{54}$$

$$\int_0^T \xi_i(t) \xi_k(t)\, dt = \delta_{ik}. \tag{55}$$

Eqs. (52) and (53) are formal expressions. Where $Y_2(t) \equiv 0$ in (3), $F(t_1, t_2)$ may not be a function in the ordinary meaning. $F(t_1, t_2)$ has been shown to be the kernel function of the linear least-squares estimator without consideration of cost [3], [8]. On the other hand, $D(t_1, t_2)$ belongs to $L_2(I_1 \times I_1)$.

Corresponding to γ_k, let us define $w_k(t)$ formally by

$$w_k(T_2 - t) = \lim_{N \to \infty} w_k^{(N)}(T_2 - t) = \lim_{N \to \infty} \sum_{n=1}^{N} \gamma_{kn} \psi_n(t). \tag{56}$$

While $w_k^{(N)}$ is a continuous function, $w_k(t)$ may not be an ordinary function where $Y_2 \equiv 0$. Consider $Y_2 \not\equiv 0$; *i.e.*, $\sigma \neq 0$. Then (15) shows that

$$\sigma_m \geq \sigma > 0.$$

From (22), it follows that $\gamma_k \in H_f$ implies $w_k \in L_2(I_2)$. According to (19), the expression below holds.

$$\Gamma_k = \lim_{N \to \infty} \int_{T_1}^{T_2} w_k^{(N)}(T_2 - t) Y(t)\, dt. \tag{57}$$[10]

Now, (56) implies that for H_f the basis $\{e_k\}$ is used in place of $\{e_k'\}$. When the basis $\{e_k\}$ is used, $\bar{\xi}_k$ is expressed as

$$\bar{\xi}_k = \lambda_k^{-1/2} V^{-1} W^T \xi_k. \tag{58}$$

Thus, $w_{k0}(T_2 - t)$ corresponding to $\lambda_k^{1/2} \cdot \bar{\zeta}_k$ is given by

$$w_{k0}(T_2 - t) = \int_0^T F(t, \tau) \xi_k(\tau)\, d\tau. \tag{59}$$

Consequently, the following corollary is easily obtained from the theorem in Section IV.

Corollary: The optimum linear finite-dimensional estimate $\tilde{X}_{\text{opt}}(t)$ is given by

$$\tilde{X}_{\text{opt}}(t) = \sum_{k=1}^{M\text{opt}} \xi_k(t) \cdot \int_{T_1}^{T_2} w_{k0}(T_2 - t) Y(t)\, dt, \tag{60}$$

where M_{opt} is defined in the theorem in Section IV. The minimum risk $r_{\min}$ is given by

$$r_{\min} = c(M_{\text{opt}}) + \int_0^T K_s(t, t)\, dt - l \sum_{k=1}^{M\text{opt}} \lambda_k.$$

Remark: From (44) and (58), we have

$$\begin{aligned} W^T W \lambda_k^{1/2} \bar{\zeta}_k &= W^T W \cdot V^{-1} W^T \xi_k \\ &= \lambda_k V V^{-1} W^T \xi_k \\ &= \lambda_k V \lambda_k^{1/2} \bar{\zeta}_k; \end{aligned}$$

i.e.,

$$\begin{aligned} &\int_0^T \int_{T_1}^{T_2} K_M(t_1, t) K_M(t_1, t_2) w_{k0}(T_2 - t_2)\, dt_1\, dt_2 \\ &\qquad = \lambda_k \int_{T_1}^{T_2} K_i(t, t_1) w_{k0}(T_2 - t_1)\, dt_1. \end{aligned} \tag{61}$$

VI. Discussion and Examples

As mentioned before, if $Y_2 \equiv 0$, $w_{k0}(t)$ does not always belong to L_2, and $w_{k0}(t)$ may involve $\delta(x - x_0)$, $\delta'(x - x_0)$, etc. Sometimes $\xi_k(t)$ and $w_{k0}(t)$ can be obtained explicitly; also, in some cases, numerical analysis is the only way of finding $\xi_k(t)$ and $w_{k0}(t)$. The procedure of employing the series $w_k^{(N)} = \sum_{k=1}^{N} \gamma_{kn} \psi_n(t)$ shown in (56) is one of those applicable in the present case. As shown in (57), with an appropriate N,

$$\int_{T_1}^{T_2} w_k^{(N)}(T_2 - t) Y(t)\, dt$$

approximates to Γ_k within any given quadratic mean error.

[10] Instead of the right-hand side of (57), let us write simply

$$\int_{T_1}^{T_2} w_k(T_2 - t) Y(t)\, dt.$$

When calculating for system design, N is preferably large enough so that the quadratic mean error due to truncation will be covered by round-off error. In the next stage, *i.e.*, circuit design, approximation should be carried out in as efficient a manner as possible.

Since T_2 and T_1 are assumed to be specified, Γ_k can be realized by a set of function-generator, multiplier, and integrator. It is necessary here to consider the cost in association with accuracy. Γ_k can also be approximately obtained by the output at T_2 of a linear filter composed of a finite number of lumped constant elements. In this case, we need to find such an appropriate real function $w_{k0}^*(t)$ as [19]:

$$w_{k0}^*(t) = \sum_{i=1}^{p} R_i e^{-s_i t}, \qquad \mathrm{Re}\, s_i > 0, \tag{62}$$

and at the same time

$$\int_{T_1}^{T_2} w_{k0}^*(T_2 - t) Y(t)\, dt$$

approximates to

$$\int_{T_1}^{T_2} w_{k0}(T_2 - t) Y(t)\, dt$$

on the average.

Here, the choice of a suitable set of $\{s_i\}$ is essential for getting good approximation with relatively few terms. The cost of the filter depends upon p, R_i's, s_i's and the required accuracy of elements. Now, let $r^{(k)}$ denote the sum of the cost of this filter and the mean loss due to error derived from approximation. Let us consider the case where $w_{k0}(t)$ is obtained in the form of $\sum \gamma_{kn}\psi_n$. Even if $N > N'$, the realization cost of

$$\int_{T_1}^{T_2} w_{k0}^{(N)}(T_2 - t) Y(t)\, dt$$

is not always higher than that of

$$\int_{T_1}^{T_2} w_{k0}^{(N')}(T_2 - t) Y(t)\, dt$$

with the same accuracy. In general, there is no relation between the basis $\{\psi_n\}$ and the cost of filter. Even though ψ_n is in the form of

$$\sum_{i=1}^{p_n} R_{ni} e^{-s_{ni} t}, \qquad \mathrm{Re}\, s_{ni} > 0, \tag{63}$$

there is not always an explicit relationship. For an illustration, consider

$$\begin{aligned} w_{k0}(t) &= e^{-st} \qquad (s \neq s_{ni}) \\ &= \lim_{N\to\infty} w_{k0}^{(N)}(t). \end{aligned}$$

If s is a moderate value, the realization cost of

$$\int_{T_1}^{T_2} e^{-s(T_2 - t)} Y(t)\, dt$$

is lower than that of

$$\int_{T_1}^{T_2} w_{k0}^{(N)}(T_2 - t) Y(t)\, dt.$$

Thus, it is difficult to formulate the relation between $r^{(k)}$ and w_{k0} to make it applicable to general instances.

For a fixed M, let us define $\tilde{X}_M$ as

$$\tilde{X}_M = \sum_{k=1}^{M} \xi_k(t) \int_{T_1}^{T_2} w_{k0}(T_2 - \tau) Y(\tau)\, d\tau, \tag{64}$$

where $\xi_k(t)$ and $w_{k0}(t)$ are defined by (54) and (59), respectively. Then, $\tilde{X}_M$ yields the minimum value of $E \,||\, X - \tilde{X} \,||^2$. (Refer to proof of theorem in Appendix.) Actually, it will be impossible to realize precisely $\tilde{X}_M$. Let $\tilde{X}_{r,M}$ denote the operator which indicates the functioning of real equipment, and let us assume that $\tilde{X}_{r,M}$ is expressed as

$$\tilde{X}_{r,M} = \sum_{k=1}^{M} \xi_k^*(t) \int_{T_1}^{T_2} w_{k0}^*(T_2 - \tau) Y(\tau)\, d\tau, \tag{65}$$

while denoting the cost of this equipment by c_M. Also, let $\tilde{X}_\xi$ and $\tilde{X}_w$ be defined as

$$\begin{aligned} \tilde{X}_\xi &= \sum_{k=1}^{M} \xi_k^*(t) \int_{T_1}^{T_2} w_{k0}(T_2 - \tau) Y(\tau)\, d\tau, \\ \tilde{X}_w &= \sum_{k=1}^{M} \xi_k(t) \int_{T_1}^{T_2} w_{k0}^*(T_2 - \tau) Y(\tau)\, d\tau. \end{aligned} \tag{66}$$

If $\xi_k^*(t)$ and $w_{k0}^*(t)$ approximate to $\xi_k(t)$ and $w_{k0}(t)$, respectively, by neglecting infinitesimal terms of higher order we can obtain:

$$\begin{aligned} &E \,||\, X - \tilde{X}_{r,M} \,||^2 - E \,||\, X - \tilde{X}_M \,||^2 \\ &\quad = E \,||\, X - \tilde{X}_\xi \,||^2 - E \,||\, X - \tilde{X}_M \,||^2 \\ &\qquad + E \,||\, X - \tilde{X}_w \,||^2 - E \,||\, X - \tilde{X}_M \,||^2. \end{aligned} \tag{67}$$

Since ξ_k and w_{k0} minimize $E \,||\, X - \tilde{X} \,||^2$,

$$\begin{aligned} &E \,||\, X - \tilde{X}_\xi \,||^2 - E \,||\, X - \tilde{X}_M \,||^2 \\ &\qquad = E \,||\, \tilde{X}_\xi - \tilde{X}_M \,||^2, \end{aligned} \tag{68}$$

$$\begin{aligned} &E \,||\, X - \tilde{X}_w \,||^2 - E \,||\, X - \tilde{X}_M \,||^2 \\ &\qquad = E \,||\, \tilde{X}_w - \tilde{X}_M \,||^2. \end{aligned} \tag{69}$$

[For validity of (68), refer to the proof of Lemma 1. Notice that the sum of the first and second terms is equal to $E \,||\, X - \tilde{X}_M \,||^2$, and the third one is equal to $E \,||\, \tilde{X}_\xi - \tilde{X}_M \,||^2$ in the right-hand side of (85). Also, the validity of (69) is assured in a similar way.]

The objective of circuit design is to minimize

$$r_m = c_M + lE \,||\, \tilde{X}_\xi - \tilde{X}_M \,||^2 + lE \,||\, \tilde{X}_w - \tilde{X}_M \,||^2. \tag{70}$$

Strictly speaking, the minimum value of r_M depends not only on M, but also on $\xi_k(t)$'s and $w_{k0}(t)$'s; still, it is almost impossible, as seen in the foregoing, to formulate this dependence. On the other hand, it is clear that min r_M usually depends mostly upon M. Hence, as a first-

order approximation, it is logical to assume that min r_M depends only on M. Then, we put

$$\min r_M = c(M).$$

Thus we can define $c(M)$ more precisely.

To summarize the foregoing, the author takes it as appropriate to consider the problem at issue in two different steps. 1) System design: find M_{opt}, $\tilde{X}_{opt}$, $\xi_k(t)$'s and $w_{k0}(t)$'s; 2) circuit design: choose the set of $\xi_k^*(t)$'s and $w_k^*(t)$'s which approximate to $\xi_k(t)$'s and $w_{k0}(t)$'s, respectively, and minimize $r_{M_{opt}}$.

In most cases, it is almost impossible to solve both steps on the same level. In step 1), consequently, step 2) is considered as represented simply by $c(M)$. If $c(M)$ cannot be defined explicitly, M_{opt} is to be decided by regarding $c(M)$ as an unknown parameter and actually estimating min r_M, M being a number of figures within some suitable range.

The object of this paper is to present a procedure for finding a solution on the level of step 1).

As an example, we shall compare the optimum estimate in Sections IV and V with the estimate of the signal waveform such as is obtained by an appropriate linear interpolation based on $\tilde{X}_\infty(t_{1k})$, the linear least-square estimates of $X(t_{1k})$ at M representative points t_{1k} of I_1. Here, $\tilde{X}_\infty(t_{1k})$ is given by

$$\tilde{X}_\infty(t_{1k}) = \int_{T_1}^{T_2} F(t_{1k}, t) Y(t)\, dt. \tag{71}$$

We may assume that

$$E \,||\, \tilde{X}_\infty(t_{1k}) \,||^2 < \infty .$$

Let us denote the above estimate by $\tilde{X}_t(t)$. Then $\tilde{X}_t(t)$ is clearly a linear finite-dimensional estimate in the meaning of Section III. We see that $\tilde{X}_t$ cannot yield less mean-squared error than $\tilde{X}_M$ defined by (64).

Next, let us consider a simple case in which the signal is contaminated by the addition of white Gaussian noise statistically independent of the signal, *i.e.*, in (3)

$$Y(t) = X(t) + Y_2(t), \tag{72}$$

and where the interval I_1 is identical with the interval I_2. In this case, the following equations are readily derived:

$$\begin{aligned} &V = U + \sigma E \ (E\text{: unit matrix}), \\ &W = U, \\ &WV^{-1}W^T = U(U + \sigma)^{-1}U. \end{aligned} \tag{73}$$

Hence, we have

$$\phi_k(t) = \psi_k(t) = \xi_k(t), \tag{74}$$

$$\lambda_k = \frac{\mu_k^2}{\mu_k + \sigma}. \tag{75}$$

Also,

$$\lambda_k^{1/2}\zeta_k = V^{-1}W^T\xi_k = \frac{\mu_k}{\mu_k + \sigma}\phi_k. \tag{76}$$

Consequently, it follows from the corollary in section V that $\tilde{X}_{opt}$ and r_{min} are given by

$$\tilde{X}_{opt}(t) = \sum_{k=1}^{M_{opt}} \frac{\mu_k}{\mu_k + \sigma}\phi_k(t) \int_0^T \phi_k(\tau) Y(\tau)\, d\tau, \tag{77}$$

$$r_{min} = c(M_{opt}) + l\int_0^T K_s(t, t)\, dt - l\sum_{k=1}^{M_{opt}} \frac{\mu_k^2}{\mu_k + \sigma}, \tag{78}$$

where

$$\int_0^T K_s(t, \tau)\phi_k(\tau)\, d\tau = \mu_k\phi_k(t). \tag{79}$$

Particularly, if

$$K_s(t, \tau) = \sum_{k=1}^{N} B_k \cdot e^{-\beta_k|t-\tau|}, \qquad (\beta_k > 0),$$

the solution of eigenvalue problem (79) has been found [15], and the realization of filters is rather simple. For example, the covariance function $K_s(t, \tau)$ is given by

$$K_s(t, \tau) = Be^{-\beta|t-\tau|} (B, \beta > 0).$$

Middleton [17] shows that

$$\mu_k = \frac{2B}{\beta(1 + q_k^2)}, \tag{80}$$

$$\phi_k(t) = C_k \cos \beta q_k\left(t - \frac{T}{2}\right) + D_k \sin \beta q_k\left(t - \frac{T}{2}\right),$$

where q_k is a positive root of:

$$\tan \beta T q_k = \frac{-2q_k}{1 - q_k^2}, \tag{81}$$

$q_1, q_2 \cdots$ are arranged in order of increasing magnitude, and C_k, D_k are constants.

Suppose that $\beta T \sim 1$ or $\beta T > 1$ and that $BT \sim \sigma$ or $BT < \sigma$. Then from (80) and (81), it follows that

$$q_k \approx \frac{k\pi}{\beta T},$$

$$\mu_k \ll \sigma,$$

except for the first few k's. Therefore except for these first k's,

$$\lambda_k = \frac{\mu_k^2}{\mu_k + \sigma} \approx \frac{4B^2\beta^2T^4}{\sigma(k\pi)^4}$$

Then a small number of filters is probably sufficient.

It will be noted that (even if there is not noise) finite-dimensional estimators cannot produce accurate signal waveforms. If $E \mid X(t) - Y(t) \mid^2 < \infty$, *i.e.*, $Y_2 \equiv 0$, to circumvent this difficulty we may apply the results in Sections IV and V to the estimation of $X(t) - Y(t)$ instead of $X(t)$ and adopt an identity operator plus a finite-dimensional operator for an estimator.

VII. Conclusion

This paper has described a new approach to the estimation of signal waveforms based on received data in a specified finite interval. The extensions to more complicated cost functions or to nonlinear finite-dimensional operators remain to be done.[11]

Appendix

Proof of Lemma 1

Let us put

$$\Phi_k = \sum_n \alpha_{kn}\phi_n, \tag{82}$$

$$\tilde{X} = \sum_n \tilde{x}_n\phi_n; \tag{83}$$

then from (17) we have

$$\tilde{x}_n = \sum_{k=1}^{M} \Gamma_k\alpha_{kn}. \tag{84}$$

From (30)

$$EL = lE \mid\mid X - \tilde{X} \mid\mid^2 = l \sum_n E(x_n - \tilde{x}_n)^2.$$

By substituting (84) into the above equation and using $E\Gamma_i\Gamma_k = \delta_{ik}$, we obtain

$$EL = l \sum_n E\left(x_n - \sum_{k=1}^{M} \Gamma_k\alpha_{kn}\right)^2$$

$$= l \sum_n \mu_n - l \sum_{k=1}^{M} \sum_n E^2(\Gamma_k x_n)$$

$$+ l \sum_{k=1}^{M} \sum_n [\alpha_{kn} - E(\Gamma_k x_n)]^2. \tag{85}$$

The above expression indicates that α_{kn} minimizing EL is given by

$$\alpha_{kn} = E(\Gamma_k x_n) \tag{86}$$

Substitute $\Gamma_k = \sum_n \sigma_n^{1/2}\gamma'_{kn}\, y_n$ and $Ey_m x_n = \rho_{nm}$ into (86); then we have

$$\alpha_{kn} = E\{\sum \gamma'_{km}\sigma_m^{-1/2} y_m x_n\}$$

$$= \sum_m \rho_{nm}\sigma_m^{-1/2}\gamma'_{km}. \tag{87}$$

This expression implies that

$$\Phi_k = WV^{-1/2}\gamma_k. \tag{88}$$

In this case, it follows from (85) and (86) that

$$EL = l \sum_n \mu_n - l \sum_{k=1}^{M} \sum_n \alpha_{kn}^2$$

$$= l \sum_n \mu_n - l \sum_{k=1}^{M} \mid\mid \Phi_k \mid\mid^2. \tag{89}$$

[11] Reference [1] also discusses minimax solutions.

Since $EL \geq 0$ and $\sum_n \mu_n < \infty$, (89) yields

$$\infty > \sum_n \mu_n \geq \sum_{k=1}^{M} \mid\mid \Phi_k \mid\mid^2.$$

This implies that

$$\Phi_k \epsilon H_s, \qquad (k = 1, 2, \cdots, M).$$

Proof of Lemma 2

Let $\bar{\xi}_k$ be defined by

$$\bar{\xi}_k = \lambda_k^{-1/2} WV^{-1/2}\zeta_k. \tag{90}$$

Using the Cauchy inequality and (39), there results

$$\mid\mid \bar{\xi}_k \mid\mid^2 \leq \lambda_k \mid\mid WV^{-1/2} \mid\mid^2 \mid\mid \zeta_k \mid\mid^2$$

$$= \lambda_k \sum_n \sum_m \rho_{nm}^2 \sigma_n^{-1} < \infty.$$

Hence

$$\bar{\xi}_k \epsilon H_s.$$

From (42), (43), and (9), it follows that

$$WV^{-1}W^T\bar{\xi}_k = \lambda_k^{-1/2} WV^{-1/2}V^{-1/2}W^T WV^{-1/2}\zeta_k$$

$$= \lambda_k^{-1/2} WV^{-1/2}\lambda_k\zeta_k$$

$$= \lambda_k\bar{\xi}_k, \tag{91}$$

$$(\bar{\xi}_i, \bar{\xi}_k) = \lambda_i^{-1/2}\lambda_k^{-1/2}\zeta_i^T V^{-1/2}W^T WV^{-1/2}\zeta_k$$

$$= \lambda_i^{-1/2}\lambda_k^{-1/2}\lambda_k(\zeta_i, \zeta_k)$$

$$= \delta_{ik}. \tag{92}$$

Now let $\bar{\zeta}_k$ be defined by

$$\bar{\zeta}_k = \lambda_k'^{-1/2} V^{-1/2}W^T\xi_k.$$

By following the same procedure as above, we obtain

$$\bar{\zeta}_k \epsilon H_f,$$

$$V^{-1/2}W^T WV^{-1/2}\bar{\zeta}_k = \lambda'_k\bar{\zeta}_k, \tag{93}$$

$$(\bar{\zeta}_i, \bar{\zeta}_k) = \delta_{ik}. \tag{94}$$

From (42), (43), (93), and (94), it follows that for any λ_i and $\lambda_k (i \neq k)$ there exists $\lambda'_{i'}$, and $\lambda'_{k'}$, such that

$$\lambda_i = \lambda'_{i'}; \qquad \lambda_k = \lambda'_{k'}, \qquad (i' \neq k').$$

Similarly, the converse relation holds. Hence

$$\lambda_k = \lambda'_k,$$

because $\{\lambda_k\}$ and $\{\lambda'_k\}$ are arranged in order of decreasing magnitude.

Proof of Theorem

Suppose that M is fixed. Lemma 1 indicates that Φ_k must be given by

$$\Phi_k = WV^{-1/2}\gamma_k$$

in order to minimize the risk. From (31) and (38) it follows that γ_k's maximizing

$$\sum_{k=1}^{M} \gamma_k^T V^{-1/2}W^T WV^{-1/2}\gamma_k$$

yield the minimum risk for fixed M. Thus, the problem is reduced to finding γ_k's that maximize $\sum_{k=1}^{M} (\gamma_k, L_f\gamma_k)$ under the restrictions $(\gamma_i, \gamma_k) = \delta_{ik}$. The solution of this problem is well known in the theory of the eigenvalue problem [14].[12] Let H_{f0} denote the eigenspace corresponding to the zero eigenvalue of L_f. (If L_f has no zero eigenvalue, H_{f0} is a null space.) γ_k's can be expressed in the following form:

$$\gamma_k = \sum_n (\gamma_k, \zeta_n)\zeta_n + \gamma_{k0}; \gamma_{k0} \in H_{f0}. \tag{95}$$

Then we obtain

$$\sum_{k=1}^{M} (\gamma_k, L_f\gamma_k) = \sum_n \lambda_n \sum_{k=1}^{M} (\gamma_k, \zeta_n)^2.$$

Here[13]

$$\sum_{k=1}^{M} (\gamma_k, \zeta_n)^2 \leq || \zeta_n ||^2 = 1,$$

$$\sum_n \sum_{k=1}^{M} (\gamma_k, \zeta_n)^2 = \sum_{k=1}^{M} || \gamma_k ||^2 = M.$$

Now let us consider the problem of minimizing

$$\sum_n \lambda_n A_n \quad (\lambda_n \geq \lambda_{n+1} > 0)$$

under the restrictions:

$$0 \leq A_n \leq 1,$$

$$\sum_n A_n = M.$$

We easily see that by letting $A_n = 1 (n = 1, 2, \cdots, M)$ and $A_n = 0 (n > M)$, we obtain the maximum value

$$\sum_{n=1}^{M} \lambda_n \quad \text{of} \quad \sum_n \lambda_n \cdot A_n.$$

From this, it follows that

$$\sum_{k=1}^{M} (\gamma_k, L_f\gamma_k) \leq \sum_{k=1}^{M} \lambda_k, \tag{96}$$

where the equality holds if

$$\sum_{k=1}^{M} (\gamma_k, \zeta_n)^2 = 1, \qquad (n = 1, 2, \cdots, M),$$

$$(\gamma_k, \zeta_n) = 0, \qquad (k = 1, 2, \cdots, M; n > M).$$

That is, $\sum_{k=1}^{M} (\gamma_k, L_f\gamma_k)$ is maximized by letting

$$\gamma_k = \zeta_k' = \sum_{i=1}^{M} t_{ki}\zeta_i, \tag{97}$$

where (t_{ki}) is an arbitrary orthogonal matrix. Here the maximum value of

$$\sum_{k=1}^{M} (\gamma_k, L_f\gamma_k) \quad \text{is} \quad \sum_{k=1}^{M} \lambda_k.$$

[12] Reference [7] treats a similar case of finite dimensions.
[13] Notice that γ_k's are orthonormal.

In case $\lambda_M = \lambda_{M+1}$, $\zeta_k (k = 1, 2, \cdots, M)$ are not uniquely determined, but any choice of ζ_k achieves the desired maximum value.

We consequently conclude that $\tilde{X}$ minimizing the risk for fixed M is given by

$$\tilde{X} = \sum_{k=1}^{M} [\zeta_k', Y] W V^{-1/2} \zeta_k'.$$

Since (t_{ik}) is an orthogonal matrix, by using (47) $\tilde{X}$ can be written as

$$\begin{aligned} \tilde{X} &= \sum_{k=1}^{M} \sum_{i=1}^{M} t_{ki}[\zeta_i, Y] \sum_{j=1}^{M} t_{kj} W V^{-1/2} \zeta_j \\ &= \sum_{i,j=1}^{M} \delta_{ij}[\zeta_i, Y] W V^{-1/2} \zeta_j \\ &= \sum_{k=1}^{M} [\zeta_k, Y] W V^{-1/2} \zeta_k = \sum_{k=1}^{M} \lambda_k^{1/2} [\zeta_k, Y] \bar{\xi}_k. \end{aligned} \tag{98}$$

By letting $\zeta_k = \bar{\zeta}_k$ in (98), we obtain

$$\tilde{X} = \sum_{k=1}^{M} \lambda_k^{1/2} [\bar{\zeta}_k, Y] \xi_k.$$

Now the risk is given by

$$r(M) = c(M) + l \sum_n^{M} \mu_n - l \sum_{k=1} \lambda_k.$$

If $r(M_1) = r(M_2)$ and $M_1 < M_2$, M_2 is preferable, because M_2 gives less average loss than M_1. Therefore the optimum M, M_{opt}, is the largest of integers that minimize $c(M) - l \sum_{k=1}^{M} \lambda_k$. Since $c(\infty) = \infty$ and $\sum_k \lambda_k < \infty$, there exists finite M_{opt}. M_{opt} satisfies the following relations:

$$l\lambda_{M_{\text{opt}}+1} \leq c(M_{\text{opt}} + 1) - c(M_{\text{opt}})$$

$$l\lambda_{M_{\text{opt}}} \geq c(M_{\text{opt}}) - c(M_{\text{opt}} - 1).$$

Suppose that

$$c(M + 1) - c(M) \geq c(M) - c(M - 1); \quad (M = 1, 2, \cdots).$$

Then, since $\lambda_M \geq \lambda_{M+1} > 0$, we easily see that M_{opt} is the largest of integers satisfying (51).

Acknowledgment

The author wishes to acknowledge the encouragement and many helpful suggestions of Prof. S. Kumagai, Associate Profs. H. Ozaki and K. Miyawaki, Department of Communication Engineering, Osaka University, Osaka, Japan, and Dr. H. Mine, Department of Electrical Engineering, Defense Academy, Kanagawa, Japan.

References

[1] T. Kasami, "On the Estimation of Waveform," Rept. of the PGIT of the Inst. Elec. Commun. Engrs., Japan; January, 1960. (In Japanese.)
[2] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's theory of prediction," *J. Appl. Phys.*, vol. 21, pp. 645–655; July, 1950.
[3] R. C. Booton, Jr., "An optimization theory for time-varying linear systems with non-stationary statistical inputs," Proc. IRE., vol. 40, pp. 977–981; August, 1952.

[4] R. C. Davis, "On the theory of prediction of non-stationary stochastic processes," *J. Appl. Phys.*, vol. 23, pp. 1047–1052; September, 1952.
[5] ——, "On the detection of sure signals in noise," *J. Appl. Phys.*, vol. 25, pp. 76–82; January, 1954.
[6] D. Van Meter and D. Middleton, "Modern statistical approaches to reception in communication theory," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 119–145; September, 1954.
[7] H. P. Kramer and M. V. Mathews, "A linear coding for transmitting a set of correlated signals," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 41–46; September, 1956.
[8] J. S. Bendat, "Exact integral equation sulutions and synthesis for a large class of optimum time variable linear filters," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 71–80; March, 1957.
[9] D. Middleton, "A note on the estimation of signal waveform," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 86–89; June, 1959.
[10] S. Darlington, "Nonstationary smoothing and prediction using network theory concepts," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 1–13; May, 1959.
[11] A. Wald, "Statistical Decision Functions," John Wiley and Sons, Inc., New York, N. Y.; 1950.
[12] M. Loeve, "Probability Theory Foundations, Random Sequences," D. Van Nostrand, Inc., Princeton, N. J., pp. 477–478; 1955.
[13] N. Wiener, "Nonlinear Problems in Random Theory," Technology Press, Cambridge, Mass., pp. 16–27; 1958.
[14] R. Courant and D. Hilbert, "Methods of Mathematical Physics I," Interscience Publishers, New York, N. Y., ch. 3; 1953.
[15] D. C. Youla, "The solution of a homogeneous Wiener-Hopf Integral equation occurring in the expansion of second-order stationary random functions," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 187–193; September, 1957.
[16] D. Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," Nachrichten d. k. Gesellschaft d. Wiss. zu Göttingen; Abschnit 4, 5, 1906.
[17] D. Middleton, "On the detection of stochastic signals in additive normal noise—part I," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 86–121; June, 1957.
[18] B. Friedman, "Principles and Techniques of Applied Mathematics," John Wiley and Sons, Inc., New York, N. Y.; p. 27, 1957.
[19] J. D. Brule, "Improving the approximation to a prescribed time response," IRE TRANS. ON CIRCUIT THEORY, vol. CT-6, pp. 355–361; December, 1959.

A Method of Digital Signalling in the Presence of Additive Gaussian Noise*

LUDWIK KURZ†, MEMBER, IRE

Summary—**This paper considers the basic problem of transmitting digital information through a noisy channel with minimum probability of error in finite time. The transmitted signals are average-power limited, and the noise is assumed to be additive Gaussian with a power spectrum which may be nonwhite. A theory of so-called efficient codes (minimax, equal separation, and nearly equal separation) is developed. Efficient codes are formed from weighted sums of eigenfunctions generated by an integral equation with its kernel corresponding to the inverse Fourier transform of the Gaussian noise power spectrum. In addition, the theory of equidistant and nearly equidistant codes [1] is extended to the case of nonwhite Gaussian noise. It is shown that efficient codes perform better than equidistant codes if the noise is nonwhite; *i.e.*, properly chosen waveforms are more efficient than binary coding. Performance results are given for several different codes when the interference is white Gaussian noise and when the noise power density increases with increasing frequency. The detection scheme used does not require estimation of the signal or the noise levels at the receiver and is thus independent of fading.**

* Received by the PGIT, May 2, 1961. The research reported in this paper was performed at New York University and was supported in part by the Natl. Science Foundation Science Faculty Fellowship and in part by the AF Command and Control Dev. Div., ARDC, Bedford, Mass., under Contract No. AF 19(604)-6168. It is based on parts of a thesis accepted by the faculty of the Graduate Div. of the College of Engrg. of New York University in partial fulfillment of the requirements for the degree of Doctor of Engineering Science.
† Research Division, College of Engrg., New York University, New York, N. Y.

INTRODUCTION

THE problem of detecting the presence or absence of fixed signals transmitted through a channel disturbed by additive Gaussian noise has received wide attention [2], [8], [9], [15]. So far, few explicit results are available concerning the proper choice of signal forms to be used in minimizing error probability if digital, but not necessarily binary, communication is to be effected in finite time through a channel disturbed by additive nonwhite Gaussian noise so as to minimize the error probability subject to the average-power limitation of the transmitter.

The demodulated noise power spectrum is assumed to increase with increasing frequency. The latter assumption is justified by the following considerations. From one point of view, any practical transmitting station transmits in an allocated band of frequencies (channel). In the middle of this band the Gaussian noise has a flat power spectrum, while near the edges of the band the noise power spectrum rises due to adjacent channel interference. From another point of view, the assumed form of the Gaussian noise power spectrum acts as a weighting function which confines the signal to an available band of frequencies.

The theory presented in this paper is an extension of the theory developed by Chang, *et al.* [1], to combat additive white Gaussian noise in the channel. Several coding procedures are discussed and the associated optimum detectors necessary to minimize the error probability of a unidirectional single-link communication system disturbed by additive nonwhite Gaussian noise, subject to average power limitation of the transmitter, are found. The main purpose of this paper, however, is to prepare background material for solution of a more important problem in digital communication—the optimization of a single-link unidirectional communication system in the presence of additive Gaussian and impulsive noise [7]. The latter theory will be presented in a later paper.[1]

For the purpose of analysis, synchronous detection of the RF signals will be presumed; the advantage of this approach is that the problem can be analyzed at video instead of RF. It will be assumed throughout the paper that the noise bandwidth is infinite, since any necessary filtering will be automatically included in the signal handling at the receiver. As a matter of fact, if the noise bandwidth were finite, one could always find a set of transmittible signals for theoretical zero probability of error.[2] No restriction is placed on the bandwidth of the signals; however, with the increase in rate of transmitted information (smaller transmission time or more complicated waveshapes), larger signal bandwidths are required.

The basic assumption in this paper of detection independent of fading is justified by the simplicity and practicability of the associated optimum receiver, and also because the resulting symmetric channel has a smaller error probability than the asymmetric channel for a fixed channel capacity [11].

The Optimum Detector

Consider a unidirectional single-link communication system. One can send through it a message of m bits by transmitting with equal *a priori* probability one of a set of time-limited signals $\{s_i(t)\}$, $0 \leq t \leq T$, $i = 1, 2, 3, \cdots, 2^m$. Corresponding to the transmission of some particular $s_i(t)$, a signal $y(t)$ is received. In general, the receiver cannot, in finite time T, determine with zero error probability to which particular transmitted signal the received signal $y(t)$ corresponds, because of the additive Gaussian noise in the channel [3], [4]. However, any receiver which retains the conditional probabilities $p(s_i/y)$ that $s_i(t)$ was transmitted when $y(t)$ is received, will preserve all the information of the received signal [14]. When a decision must be made, and only one signal can be selected out of 2^m possible signals, the best a receiver can do is select the signal with the greatest *a posteriori* probability $p(s_i/y)$ [14]. This receiver will be considered optimum.

To determine the conditional probability $p(s_i/y)$, expand both $y(t)$ and $s_i(t)$ into a generalized Fourier series [2] using the orthonormal set $\{\varphi_k(t)\}$, generated by the integral equation

$$\sigma_k^2 \varphi_k(t) = \int_0^T K(x - t)\varphi_k(x)\, dx, \qquad 0 \leq x \leq T \tag{1}$$

where $K(x)$ is the inverse Fourier transform of the noise power spectrum. Then

$$s_i(t) = \sum_{k=1}^{N} a_{ik}\varphi_k(t), \tag{2}$$

where

$$a_{ik} = \int_0^T s_i(t)\varphi_k(t)\, dt$$

and

$$\begin{aligned} y(t) &= n(t) + s_i(t) \\ &= \sum_{k=1}^{N} y_k \varphi_k(t), \end{aligned} \tag{3}$$

where

$$y_k = n_k + a_{ik} \tag{4}$$

and

$$n_k = \int_0^T n(t)\varphi_k(t)\, dt. \tag{5}$$

By the Karhunen-Loéve theorem [6], all n_k are Gaussian and statistically independent with variances σ_k^2 and means zero; therefore, all y_k are Gaussian and statistically independent with variances σ_k^2 and means a_{ik}.

Since the decision at the receiver is made so that the signal with the largest *a posteriori* probability is accepted, the test reduces to choosing $s_i(t)$ among a pair of signals $s_i(t)$ and $s_j(t)$, if

$$\sum_{k=1}^{N} \frac{y_k(a_{ik} - a_{jk})}{\sigma_k^2} \geq 0 \tag{6}$$

and $s_j(t)$ otherwise[3] assuming detection independent of fading, namely,

$$a_{ik} = \pm a_{jk}. \tag{7}$$

Thus, the optimum receiver must determine

$$A_i = \sum_{k=1}^{N} \frac{y_k a_{ik}}{\sigma_k^2} \quad \text{for all} \quad i = 1, 2, 3, \cdots, 2^m$$

and accept the signal $s_i(t)$ which produces the largest A_i. A_i can easily be formed by crosscorrelating $y(t)$ with $\sum_{k=1}^{N} [a_{ik}/\sigma_k^2]\varphi_k(t)$, which are known to the receiver. Thus, our optimum receiver is just a simple crosscorrelator between received signals and signals available at the receiver.

[1] See also [18] and [19].
[2] Transmit the signals in the noise-free band of frequencies.
[3] See [17] or [19].

Detection Error, Separation Function, Efficient Codes

Let P_{ij} denote the error probability that signal $s_j(t)$ was accepted when signal $s_i(t)$ had been sent excluding all the other signals. We shall call this error probability the detection error. This error can be expressed as

$$P_{ij} = \frac{1}{2}\left[1 - \Phi\left(\frac{\xi_{ij}}{\sqrt{2}}\right)\right], \tag{8}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz \tag{9}$$

which is the tabulated error integral, and

$$\xi_{ij} = \frac{\bar{v}_{ij}}{\sigma_v}, \qquad \bar{v}_{ij} = \sum_{k=1}^{N} \frac{a_{ik}(a_{ik} - a_{jk})}{\sigma_k^2},$$

$$\sigma_v^2 = \sum_{k=1}^{N} \frac{(a_{ik} - a_{jk})^2}{\sigma_k^2}. \tag{10}$$

The assumption of detection independent of fading results also in a symmetric operation of the system, namely,

$$P_{ij} = P_{ji}. \tag{11}$$

Since a symmetric channel with a given capacity has a smaller error probability then an asymmetric channel, with the same capacity, unless the capacity of the channel is very low [11],[4] or the choice of detection independent of fading results not only in a simple optimum receiver, but assures also a symmetric channel with its associated small error probability.

The condition expressed by (7), which assures both fading independent detection and a symmetric channel, has the following physical interpretation. Any signal $s_i(t)$ may be considered as consisting of a sum of orthogonal digits[5] independently affected by noise. A single such digit may be represented by

$$d_{i\mu} = a_{i\mu}\varphi_\mu(t). \tag{12}$$

Since noise affects the digit through its energy and is insensitive to the polarity, the digit $d_{i\mu}$ and $-d_{i\mu}$ will be affected by noise in the same manner. Eq. (6) implies equal cost to the receiver of error in signal $s_i(t)$ and $s_j(t)$, and this in turn signifies that every corresponding orthogonal digit of signal $s_i(t)$ and signal $s_j(t)$ is affected by noise in the same manner.

Thus, for a system with fading independent detection, each signal of the set $\{s_i(t)\}$, $i = 1, 2, 3, \cdots, 2^m$ must have the same number of orthogonal digits, u; for the same index of the orthogonal digit, $k = \mu$, the digits must be equal in magnitude and be the same in form but can be opposite in sign. For any two signals $s_i(t)$ and $s_j(t)$ of the set, λ digits will be of opposite sign and $(u - \lambda)$ digits will be of the same sign. The function ξ_{ij}^2 will reduce to

$$\xi_{ij}^2 = \sum_k \frac{a_{ik}^2}{\sigma_k^2}, \tag{13}$$

where the summation is to be taken over those k for which $a_{ik} = -a_{jk}$.

The larger ξ_{ij}^2, the smaller the detection error, P_{ij}; therefore, it is logical to call ξ_{ij}^2 the separation function. Any code having the separation function of the form (13) and consisting of signals formed from sums of orthogonal digits is called an efficient code.

Methods of Coding and the Associated Error Probability

It is desirable to select the set of signals $\{s_i(t)\}$ to minimize the error probability P_e, subject to the average-power limitation of the transmitter. The latter can be expressed mathematically as

$$\int_0^T s_i^2(t)\, dt = \sum_{k=1}^{N} a_{ik}^2 = S^2T \tag{14}$$

$$\text{for} \quad i = 1, 2, \cdots, 2^m.$$

In the simple case of one bit of information ($m = 1$)

$$P_e = P_{12} = P_{21} = \frac{1}{2}\left[1 - \Phi\left(\frac{\xi_{12}}{\sqrt{2}}\right)\right], \tag{15}$$

where $\xi_{12}^2 = S^2T/\sigma_1^2$, $s_1(t) = -s_2(t) = S\sqrt{T}\varphi_1(t)$, and $\varphi_1(t)$ is the eigenfunction of the integral equation (1) corresponding to the lowest eigenvalue σ_1^2.

If we attempt to develop a similar coding procedure for $m > 1$, we run into some difficulties. The exact expression for P_e cannot be found unless the set of signals $\{s_i(t)\}$ is given, and if we satisfy ourselves with minimization of the pessimistic expression[6]

$$P_e \leq 1 - 2^{-m} \sum_{i=1}^{2^m} \prod_{\substack{j=1 \\ i \neq j}}^{2^m} (1 - P_{ij}), \tag{16}$$

the problem is still too difficult to handle if one considers the functional form of P_{ij}.

Thus, we are forced to develop several efficient codes which are optimum in some sense and then compare their performance.

The Minimax Code

The minimax code is an efficient code which satisfies the following conditions:

[4] The proof in Silverman's paper is for the binary channel, but the same proof is valid for nonbinary channels if the correlation among detection errors, P_{ij}, is negligible.

[5] The term "orthogonal digit" is used to differentiate from the usually used term binary digit. It is defined by (12).

[6] See [1] or [10].

1) Every signal of the set $\{s_i(t)\}$ is composed of a minimum number of orthogonal digits.
2) The waveforms of the orthogonal digits are the ordered eigenfunctions of (1). The eigenfunctions are selected in increasing order, starting from the lowest, until as many as necessary have been chosen.
3) The largest detection error P_{ij} is minimized by proper selection of the coefficients $\{a_{ik}\}$.

Conditions 1) and 2) assure least interaction between signal and noise in the channel. Condition 3) assures that the signals are resistant to the most adverse action of the channel and gives rise to the name of this code, minimax, or the code which minimizes the maximum detection error.

If m bits of information are to be transmitted, the minimum number of orthogonal digits in each signal is m, or $u = m$. The maximum detection error will occur for any two signals $s_i(t)$ and $s_j(t)$ when they differ by one orthogonal digit. Since this error must be minimized subject to average-power limitation of the signal, it means that every digit of any signal $s_i(t)$ must be affected by the noise in the same manner, or for any $s_i(t)$ of the signal set $\{s_i(t)\}$, $i = 1, 2, \cdots, 2^m$ the following conditions must be satisfied:

$$\frac{a_{i1}}{\sigma_1} = \frac{a_{i2}}{\sigma_2} = \cdots = \frac{a_{i\mu}}{\sigma_\mu} = \cdots = \frac{a_{im}}{\sigma_m} \tag{17}$$

and

$$\sum_{k=1}^{m} a_{ik}^2 = S^2T. \tag{18}$$

Using (17), (18), and (12), one obtains the expression for the μth orthogonal digit of the ith signal

$$d_{i\mu} = \pm a_{i\mu}\varphi_u(t), \tag{19}$$

where

$$a_{i\mu} = S\sqrt{T}\,\frac{\sigma_\mu}{\sqrt{\sum_{k=1}^{m}\sigma_k^2}}, \qquad \mu = 1, 2, 3, \cdots, m. \tag{20}$$

The set of signals $\{s_i(t)\}$ can now be found from the matrix equation

$$M_m A_m \Phi_m = \Lambda. \tag{21}$$

The matrix M_m is formed by writing all 2^m binary numbers in ascending order from zero to $2^m - 1$ and replacing every zero by -1. Obviously, the M_m matrix is a $2^m \times m$ matrix. We shall call the M_m matrix the coding matrix of the minimax code.

The A_m matrix is a square diagonal matrix with elements of it found from (20) using the following rule

$$\begin{aligned} a_{i\mu} &= 0 \quad \text{for} \quad i \neq \mu \\ &= a_{\mu\mu} \quad \text{for} \quad i = \mu. \end{aligned} \tag{22}$$

The Φ_m matrix is a column matrix consisting of m rows and the elements of the matrix are the eigenfunctions $\{\varphi_\mu(t)\}$, $\mu = 1, 2, \cdots, m$, chosen to correspond to the eigenvalues σ_μ^2.

The Λ matrix is a column matrix consisting of 2^m rows; the elements of the rows are the signals $s_i(t)$, $i = 1, 2, \cdots, 2^m$.

Using (13) and (20), the separation function for the minimax code reduces to

$$\xi_{ij}^2 = \xi_\lambda^2 = \frac{\lambda S^2 T}{\sum_{k=1}^{m}\sigma_k^2} \quad \text{for} \quad \lambda = 1, 2, \cdots, m. \tag{23}$$

The error probability for the minimax code is

$$P_e = 1 - (1 - P_{\lambda=1})^m \tag{24}$$

where $P_{\lambda=1}$ is the detection error corresponding to the separation function of (23) for $\lambda = 1$.

The Equal Separation Code

Consider an efficient code with all the separation functions ξ_{ij}^2 equal and the total separation

$$A_T = \tfrac{1}{2}\sum_{i=1}^{2^m}\sum_{\substack{j=1 \\ i\neq j}}^{2^m} \xi_{ij}^2 \tag{25}$$

at its maximum. A code defined in this manner determines a stationary point for the error probability.[7] The equal separation code will possess the following properties.

1) $P_{ij} = P_{ji}$.
2) All separation functions ξ_{ij}^2 and all detection errors P_{ij} are equal.
3) The total separation A_T is at its maximum. The separation function for this code is of the form

$$\xi_{ij}^2 = \frac{\lambda S^2 T}{\sum_{k=1}^{u}\sigma_k^2}, \tag{26}$$

where λ and u are fixed and independent of i and j. Thus, to satisfy conditions 1), 2), and 3), u and λ must be the same for all signals of the set $\{s_i(t)\}$; u must be as small as possible and λ as large as possible.[8]

The set of signals $\{s_i(t)\}$ can now be found from the matrix equation

$$M_s A_s \Phi_s = \Lambda. \tag{27}$$

Matrices A_s and Φ_s are of the same form as matrices A_m and Φ_m, only the index μ varies up to u and not m, namely, $\mu = 1, 2, \cdots, u$. The matrix remains the same as for the minimax code. It now remains to form the coding matrix M_s. This matrix must possess the following properties.

1) It must be a $2^m \times u$ matrix,
2) The number of columns, u, must be a minimum, and λ a maximum.
3) Each element of the matrix must be either a plus or a minus one.

[7] The proof of this statement is identical with the proof given for equidistant codes in Appendix B of [1].
[8] This will assure maximum separation among signals.

4) Each row must differ from any other row by the same number of elements λ and have $(u - \lambda)$ of the same elements.[9]

The M_S matrix is generated using Slepian's [13] $(2^m - 1, m)$ group alphabet.[10] The code is constructed by writing each word number and digit number in binary notation from 1 to $2^m - 1$. The first generator is then formed from the unit digits of each digit number, the second generator from the next place digits . . . etc. The other code words are determined from the generators in conventional fashion by addition modulo two of the corresponding digits.

Table I shows the generation of the code words when $m = 3$.

TABLE I

word number	Digit Number	
	0 0 0 1 1 1 1	third generator
	0 1 1 0 0 1 1	second generator
	1 0 1 0 1 0 1	first generator
0 0 1	1 0 1 0 1 0 1	first generator
0 1 0	0 1 1 0 0 1 1	second generator
0 1 1	1 1 0 0 1 1 0	
1 0 0	0 0 0 1 1 1 1	third generator
1 0 1	1 0 1 1 0 1 0	
1 1 0	0 1 1 1 1 0 0	
1 1 1	1 1 0 1 0 0 1	
	0 0 0 0 0 0 0	

The matrix M_s is now formed by taking the Slepian's $(2^m - 1, m)$ group code and replacing every zero with minus one. The resulting matrix will be a $2^m \times (2^m - 1)$ matrix in which every row differs from every other row in the matrix by 2^{m-1} elements. Eq. (26) will now reduce to

$$\xi_{ij}^2 = \xi_s^2 = \frac{2^{m-1} S^2 T}{\sum_{k=1}^{2^m - 1} \sigma_k^2}. \tag{28}$$

The corresponding expression for the error probability P_e is

$$P_e \leq 1 - (1 - P_s)^{2^m - 1} \tag{29}$$

and P_s is the detection error corresponding to the separation function (28).

The Equidistant Code

Since the codes developed so far are only optimum in a certain sense, it is advisable to investigate some other possibilities of coding. One such possibility is an extension of the theory of equidistant codes as developed by Chang, *et al.* [1], to combat additive white Gaussian noise in the channel to the case of nonwhite Gaussian noise. Consider binary digital coding, namely, coding into pulse trains consisting of identically shaped pulses both positive and negative. Since $P_{ij} = P_{ji}$ and the signal set $\{s_i(t)\}$ is average-power limited, each signal of the set must consist of the same number of binary digits, u. It is obvious that if the decision time is fixed at some value T and if there are u digits in each signal, then the pulse shape for the kth digit of the ith signal must be of the form

$$d_{ik} = \pm S \sqrt{\frac{T}{u}} \Psi_1(t), \tag{30}$$

where $\Psi_1(t)$ is eigenfunction corresponding to the lowest eigenvalue τ_1^2 of the integral equation (1) with T replaced by T/u.

The coding matrix M_D will consist of plus and minus ones; it will be a $2^m \times u$ matrix. The D matrix whose elements are the binary digits of the signal set $\{s_i(t)\}$ may be expressed as

$$D = a\Psi_1(t) M_D, \tag{31}$$

where

$$a = S \sqrt{\frac{T}{u}}. \tag{32}$$

Let us define a new function

$$D_{ij} = \frac{1}{4S^2 T} \int_0^T [s_i(t) - s_j(t)]^2 \, dt, \tag{33}$$

which is the distance between any two signals $s_i(t)$ and $s_j(t)$ of the signal set $\{s_i(t)\}$. Let the ith and jth rows of the matrix D differ by λ binary digits, then

$$D_{ij} = \frac{\lambda}{u}.$$

Consider now a system of coding for which all the distances D_{ij} are equal and the total distance D_T as defined by

$$D_T = \tfrac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{2^m} \sum_{j=1}^{2^m} D_{ij} \tag{34}$$

is at its maximum. Such code determines a stationary point for the error probability.[7] The coding matrix M_D for the equidistant code will be identical with the coding matrix for the equal separation code, or $\lambda = 2^{m-1}$ and $u = 2^m - 1$, and the M_D matrix is a Slepian $[2^m - 1, m]$ group alphabet with every zero replaced by minus one.

Using the same procedure as for orthogonal digit coding, the separation function of the equidistant code can be expressed as

$$\xi_{ij}^2 = \xi_D^2 = \frac{2^{m-1}}{(2^m - 1)} \frac{S^2 T}{\tau_1^2}. \tag{35}$$

The detection error and the corresponding error probability are found as for the equal separation code.

Other Codes

The three types of codes discussed so far do not exhaust the possible schemes of coding one can develop. For instance, one can develop nearly equidistant and nearly equal separation codes which possess certain advantages.

[9] This property of the matrix will assure equal separation property of the code.

[10] See [1] or [16], code Type 10′. A detailed discussion is given in [17] and [19].

1) *Nearly Equal Separation Codes*: The coding matrix for these codes also is formed from Slepian group alphabets, but inequality in the distances between sequences is permitted, namely,

$$\frac{\lambda}{u} \geq \frac{2^{m-1}}{2^m - 1}. \tag{36}$$

Obtained in this way, signal set $\{s_i(t)\}$ has some inequality in separation functions, but the total separation remains at a maximum value. These codes will require fewer orthogonal digits per bit of information transmitted, and their performance will be better than that of other codes for moderate information rates and moderate increase of noise power spectrum with frequency.

2) *Nearly Equidistant Codes*: The nearly equidistant codes are an extension of the equidistant code theory in the same manner as the nearly equal separation codes are an extension of the equal separation code theory. The coding matrix for the nearly equidistant codes is the same as for corresponding equal separation codes.

The nearly equidistant codes permit a reduction in the number of binary digits per bit with moderate increase in error probability P_e, which results in a reduction in the equipment cost in return for a slight increase in the error probability if the noise power spectrum is white.

If the noise power spectrum increases with frequency and the decision time remains constant, reduction in the number of binary digits per bit for the nearly equidistant codes may result in an actual decrease in the error probability.

The expression for the error probability for codes 1) and 2) is of the form

$$P_e \leq 1 - \prod_{\lambda=i}^{j} (1 - P_\lambda)^{\gamma_\lambda} \tag{37}$$

where i is the lowest and j the highest weight of a row in the coding matrix, excluding the identity vector; γ_λ is the number of rows in the coding matrix of weight λ. The constant γ_λ must satisfy

$$\sum_{\lambda=i}^{j} \gamma_\lambda = 2^m - 1. \tag{38}$$

For the nearly equal separation codes, the separation function is

$$\xi_\lambda^2 = \frac{\lambda S^2 T}{\sum_{k=1}^{u} \sigma_k^2}, \lambda = i, i+1, \cdots, j. \tag{39}$$

For the nearly equidistant codes, the separation function is

$$\xi_\lambda^2 = \frac{\lambda S^2 T}{u\tau_1^2}, \lambda = i, i+1, \cdots, j. \tag{40}$$

The detection errors P_λ are found in both cases in the usual manner.

Comparison of Various Codes

One realizes at once that to minimize the error probability for any of the codes described above, the separation functions must be made as large as possible; zero error probability can only be achieved either for $T \to \infty$ or $S^2 \to \infty$. For a finite decision time T and finite average power of the transmitter S^2, the error probability cannot be reduced to zero, which agrees with the theory developed by Shannon [12] and refined by Feinstein [4] and Elias [3].

Let us now study the influence of change in S^2 and T for various codes and different power spectra of noise.

White Gaussian Noise, $\Phi_n(j\omega) = A^2$

In this case, (1) will reduce respectively to

$$\sigma_k^2 \varphi_k(t) = A^2 \varphi_k(t) \tag{41}$$

for orthogonal-digit coding and

$$\tau_k^2 \psi_k(t) = A^2 \psi_k(t), \tag{42}$$

for binary digit coding. Thus

$$\sigma_k^2 = \tau_k^2 = A^2 \tag{43}$$

for all k. Any orthonormal set on $(0, T)$ or $(0, T/u)$ will do. This permits an infinite choice of signal forms for coding because one can use any orthonormal set in the interval $(0, T)$ or $(0, T/u)$ to generate a particular code. The choice of any specific orthonormal set will be governed mainly by the bandwidth requirements and ease of generation of the resulting signal set $\{s_i(t)\}$.

From (43), (28), and (35), one can see that in the white noise case the separation function will be the same for equidistant and equal separation codes, so that both codes are equally good in minimizing the error probability. One would probably prefer to use the equidistant code because the detector for it is quite simple.[11]

In this case, the error probability for the minimax code is the same as for the binary uncoded case. From the form of the separation functions, it is obvious that for infinite bandwidth, increase in average power S^2 and decision time T are equally successful in decreasing the error probability for all codes.

Gaussian Noise with the Power Spectrum of the Form $\Phi_n(j\omega) = A^2 + B^2\omega^2$

The eigenvalues for this type of noise are[12]

$$\sigma_k^2 = \frac{B^2 k^2 \pi^2}{T^2} + A^2 \tag{44}$$

$$\tau_1^2 = \frac{u^2 B^2 \pi^2}{T^2} + A^2. \tag{45}$$

[11] A simple detector for the equidistant code can be found in [1], p. 22; a modified version of it is shown in [18] and [19].

[12] See Appendix.

one examines the expressions for the separation functions
id (44) and (45), one can see that increase in decision
me T is much more effective than increase in average
ower of the transmitter S^2 in increasing the value of the
paration function for all codes, thus decreasing the
robability of error P_e.

Fig. 1 and 2 show the behavior of P_e as a function of
for various codes and fixed values of S^2, $\Phi_n(j\omega)$, and m.[13]
ne can conclude from these figures that for high informa-
on rate and noise power spectrum that increases rapidly
ith increasing frequency, the minimax code performs
est. On the other hand, for low information rate and noise
ower spectrum that increases slowly with increasing
equency, the equal separation code performs best.
he equidistant code is always poorer than the equal
paration code. This can be verified mathematically
y means of (28) and (35). Thus,

$$\frac{\xi_S^2}{\xi_D^2} = \frac{u\tau_1^2}{\sum_{k=1}^{u} \sigma_k^2} \geq \frac{\tau_1^2}{\sigma_k^2}. \tag{46}$$

he equality sign holds only for white Gaussian noise
$B^2 = 0$). Using (44) and (45)

$$\frac{\tau_1^2}{\sigma_u^2} = 1, \tag{47}$$

nd then from (46) and (47), it follows that

$$\xi_S^2 > \xi_D^2 \tag{48}$$

emonstrating that the equal separation code has a
rger separation function (lower error probability) than
ie equidistant code if the conditions for transmission
re the same in both cases.

If B^2 and m are small and T is large, the performance of
ie equidistant code approaches that of the equal separa-
on code. Since the detector for the equidistant code is
uite simple, it may be preferable to use the equidistant
ode for low information rate and noise power spectrum
iat increases slowly with increasing frequency.

Increase in the Error Probability P_e Due to Finite Bandwith of the System

Up to this point, it has been assumed that the bandwidth
f the system is infinite. In an actual system, every signal
(t) of the set $\{s_i(t)\}$ will not only be corrupted by addi-
ve Gaussian noise but also distorted because the band-
idth of the system is finite. This will give rise to an
ncrease in the error probability due to interdigit cross-
lk. The actual received signal, excluding the effect of
oise, is of the form

$$z_i(t) = \sum_{k=1}^{N} b_{ik}\varphi_k(t), \tag{49}$$

here $b_{ik} \neq a_{ik}$.

[13] Note that the information rate is m/T.

NOISE POWER SPECTRUM $\Phi_n(j\omega)$	AVERAGE NORMALIZED POWER AVAILABLE, S^2	TYPE OF CODE USED	NUMBER OF BITS TRANSMITTED, m	CURVE NUMBER
$0.01\omega^2 + 1$	10	MINIMAX	2	1
" "	"	EQUAL SEPARATION	2	2
" "	"	EQUIDISTANT	2	3
" "	"	MINIMAX	3	4
" "	"	EQUAL SEPARATION	3	5
" "	"	EQUIDISTANT	3	6

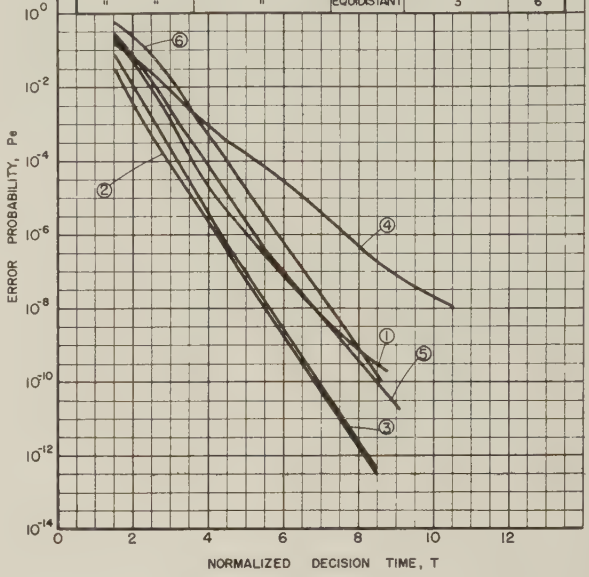

Fig. 1—Error probability as a function of normalized decision time.

NOISE POWER SPECTRUM $\Phi_n(j\omega)$	AVERAGE NORMALIZED POWER AVAILABLE, S^2	TYPE OF CODE USED	NUMBER OF BITS TRANSMITTED, m	CURVE NUMBER
$0.1\omega^2 + 1$	10	MINIMAX	2	1
" "	"	EQUAL SEPARATION	2	2
" "	"	EQUIDISTANT	2	3
" "	"	MINIMAX	3	4
" "	"	EQUAL SEPARATION	3	5
" "	"	EQUIDISTANT	3	6

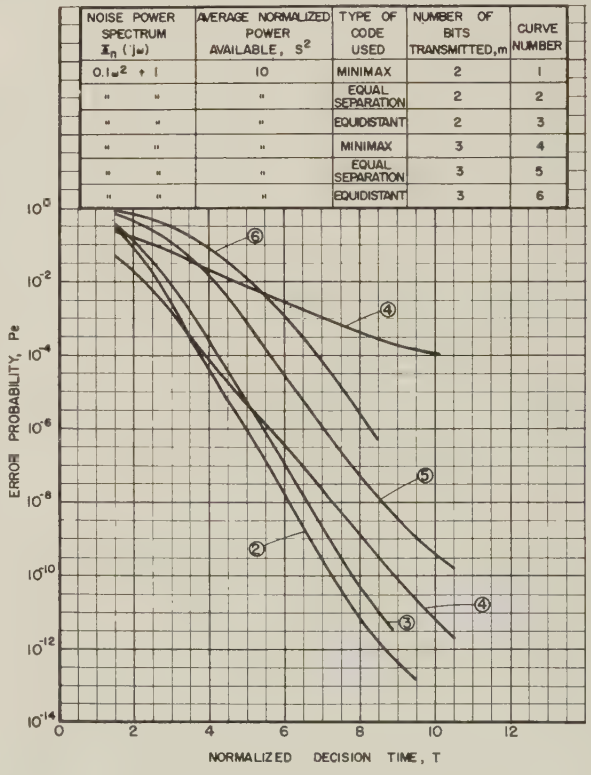

Fig. 2—Error probability as a function of normalized decision time.

If the bandwidth of the system is much larger than the bandwidth of the signal, $b_{ik} \doteq a_{ik}$, and the rise in error probability is negligible. In general, when b_{ik} differs appreciably from a_{ik}, the symmetry of the system will be disturbed and the error probability will increase. The actual error probability can easily be found by using b_{ik} instead of a_{ik} throughout the analysis.

The increase in error probability can be partially reduced if one uses at the detector signals composed of orthogonal digits with b_{ik} replacing a_{ik} for orthogonal digit coding and a pulse of the form $S\sqrt{T/u}\, f_1(t)$, where $f_1(t)$ corresponds to the distorted form of $\Psi_1(t)$ for $0 \leq t \leq T/u$. The distortion is caused by the finite bandwidth of the system.

Appendix[14]

Eigenvalues and Eigenfunctions of the Integral Equation

$$\sigma_k^2 \varphi_k(t) = \int_0^T \varphi_k(x) K(x - t)\, dx, \qquad 0 \leq x \leq T,$$

where

$$K(x) = \mathfrak{F}^{-1}[A^2 + B^2\omega^2].$$

Since the power spectrum $\Phi_n(j\omega) = A^2 + B^2\omega^2$ is given, the inverse Fourier transform of it, $K(x)$, which forms the kernel of the integral equation, may be found, namely,

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [A^2 + B^2\omega^2] e^{j\omega x}\, d\omega \tag{50}$$

$$= A^2\, \delta(x) - B^2\, \delta''(x),$$

where $\delta(x)$ is the Dirac delta function and $\delta''(x)$ its second derivative. Substituting from (50) into the integral equation, we obtain

$$B^2 \frac{d^2\varphi_k(t)}{dt^2} + (\sigma_k^2 - A^2)\varphi_k(t) = 0. \tag{51}$$

The set of functions $\{\varphi_k(t)\}$ which satisfies the differential equation (51) and the conditions

$$\int_0^T \varphi_i(t)\varphi_j(t)\, dt = \delta_{ij}, \tag{52}$$

where δ_{ij} is the Kronecker delta function, and

$$\varphi_k(t) = 0 \quad \text{for} \quad t = 0 \quad \text{or} \quad T \tag{53}$$

is the set of eigenfunctions used for coding and the corresponding σ_k^2 are the eigenvalues.

The fact that the set $\{\varphi_k(t)\}$ is a solution to (51) assures that it is a solution of the integral equation. The condition expressed by (52) assures that the set is orthonormal, and the condition expressed by (53) will keep the switching impulses to a minimum, thus decreasing the unwanted impulsive noise.

Let us rewrite (51) in the form

$$\frac{d\varphi_k(t)}{dt^2} + a_k^2\varphi_k(t) = 0 \tag{54}$$

where

$$a_k^2 = \frac{\sigma_k^2 - A^2}{B^2}. \tag{55}$$

The solutions to (54) are of the form

$$\varphi_k(t) = K_1 \cos a_k t + K_2 \sin a_k t. \tag{56}$$

For $\varphi_k(t)$ to generate a set $\{\varphi_k(t)\}$ satisfying the conditions expressed by (52) and (53), all a_k^2 must be positive and real, or (55) must be restricted by the inequality

$$\sigma_k^2 > A^2. \tag{57}$$

To satisfy the condition expressed by (53),

$$a_k = \frac{k\pi}{T},$$

$$\text{and} \quad K_1 = 0. \tag{58}$$

The corresponding eigenfunctions are

$$\varphi_k(t) = \sqrt{\frac{2}{T}} \sin \frac{k\pi}{T} t \quad \text{for} \quad k = 1, 2, \cdots, N. \tag{59}$$

Using (58) and (55), one can find the eigenvalues σ_k^2 corresponding to the eigenfunctions given by (59), namely,

$$\sigma_k^2 = \frac{k^2B^2\pi^2}{T^2} + A^2. \tag{60}$$

Let T now be reduced to T/u, and let the new eigenvalues and eigenfunctions be τ_k^2 and $\Psi_k(t)$, respectively; then (60) will transform into

$$\tau_k^2 = \frac{k^2u^2B^2\pi^2}{T^2} + A^2, \tag{61}$$

and (59) into

$$\psi_k(t) = \sqrt{\frac{2u}{T}} \sin \frac{ku\pi}{T} t \quad \text{for} \quad k = 1, 2, \cdots, N. \tag{62}$$

Acknowledgment

The author wishes to thank Dr. S. S. L. Chang for his suggestions and guidance and Prof. L. S. Schwartz, whose suggestions were of considerable help.

Bibliography

[1] S. S. L. Chang, B. Harris, A. Hauptschein, D. Hoffman, K. Morgan, and L. S. Schwartz, "Evaluation and Optimization of Digital Communication Systems," New York University, New York, N. Y., Fourth Scientific Rept., Contract AF 19(604)-1964; January, 1958.

[2] W. B. Davenport, Jr., and W. L. Root, "An Introduction to the Theory of Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., chs. 6 and 14; 1958.

[14] The procedure in this Appendix is essentially the same as in Appendix 2 of [2]. It can be justified using the theory of distributions in [20]–[23]; namely, the set of functions $\{\varphi_k(t)\}$ generated by (51) belongs to the class D (see [20]) and $K(x)$ is a symbolic function (see [21]).

3] P. Elias, "Coding for noisy channels," 1955 IRE Convention Record, pt. 4, pp. 37–46.
4] A. Feinstein, "A new basic theorem of information theory," IRE Trans. on Information Theory, vol. IT-4, pp. 2–23; September, 1954.
5] R. Filipowsky, "Recent progress in applying information theory to digital transmission systems," *Trans. AIEE, (Commun. and Electronics,* no. 40), pp. 848–855; January, 1959.
6] M. Loéve, "Probability Theory," D. Van Nostrand Co., Inc., Princeton, N. J., p. 478; 1955.
7] R. Mertz, "Model of impulsive noise for data transmission," 1960 IRE International Convention Record, pt. 5, pp. 247–259.
8] D. Middleton and D. Van Meter, "Detection and extraction of signals in noise from the point of view of statistical decision theory," *J. Soc. Ind. and Appl. Math.*, vol. 3, pp. 192–253, December, 1955; vol. 4, pp. 86–119, June, 1956.
9] W. W. Peterson, T. G. Birdsall, and W. C. Fox, "Theory of signal detectability," IRE Trans. on Information Theory, vol. IT-4, pp. 171–212; September, 1954.
0] E. N. Gilbert, "A comparison of signalling alphabets," *Bell Sys. Tech. J.*, vol. 31, pp. 504–522; March, 1952.
1] R. A. Silverman, "On binary channels and their cascades," IRE Trans. on Information Theory, vol. IT-1, pp. 19–27; December, 1955.
2] C. E. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. J.*, vol. 27, pp. 379–423, July, 1948; pp. 623–656, October.
3] D. Slepian, "A class of binary signalling alphabets," *Bell Sys. Tech. J.*, vol. 35, pp. 203–234; January, 1956.
4] P. M. Woodward, "Probability and Information Theory with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953.
5] B. Zadeh and J. R. Ragazzini, "Optimum filters for the detection of signals in noise," Proc. IRE, vol. 40, pp. 1223–1231; October, 1952.
[16] J. E. MacDonald, "Design methods for maximum minimum-distance error-correcting codes," *IBM J. Res. & Dev.*, vol. 4, pp. 43–57; January, 1960.
[17] L. Kurz, "An Optimization Procedure for a Single-Link Unidirectional Digital Communication System in the Presence of Additive Gaussian Noise and for Detection Independent of Fading," New York University, New York, N. Y., Third Scientific Rept., Contract No. AF 19(604)-6168; September, 1960.
[18] L. Kurz, "An Optimization Procedure for a Single-Link Unidirectional Digital Communication System in the Presence of Additive Gaussian and Impulsive Noise and for Detection Independent of Fading," New York University, New York, N. Y., Fourth Scientific Rept., Contract No. AF 19(604)-6168; September, 1960.
[19] L. Kurz, "An Optimization Procedure for a Single-Link Unidirectional Digital Communication System in the Presence of Additive Gaussian and Impulsive Noise and for Detection Independent of Fading," An Engrg. Sc. D. thesis, College of Engrg., New York University, New York, N. Y.; Spring, 1961.
[20] P. A. Clavier, "Some applications of the Laurent Schwartz distribution theory to network problems," *Proc. Symp. on Modern Network Synthesis,* Polytechnic Inst. of Brooklyn, N. Y., pp. 249–265; April, 1955.
[21] B. Friedman, "Lectures on Applications of Distributions," prepared for the Atomic Energy Commission, Brookhaven Natl. Lab., Summer, 1959.
[22] I. Halperin, "Introduction to the Theory of Distributions," University of Toronto Press, Toronto, Can.; 1952.
[23] L. Schwartz, "Théorie des Distributions," Herman and Co., Paris, France, vols. 1 and 2; 1950.

Some Results on the Problem of Discriminating Between Two Gaussian Processes*

P. BELLO†, ASSOCIATE MEMBER, IRE

Summary—This paper is concerned with certain aspects of the problem of discriminating between two Gaussian processes. The emphasis is on determining approximate optimum detector structures which avoid some of the mathematical difficulties inherent in the evaluation of the exact optimum detector structure. To this end, an approach termed the "inverse operator" approach is presented which leads to approximate detector structures via the Neumann series expansion of linear operator theory. These approximate detectors are found by using a finite number of terms in an "optimun detector" expansion which results from the use of the above Neumann series expansion. A sufficient condition for the rapid convergence of the optimum detector expansion is found to be that the eigenvalues of a certain operator have magnitudes much less than unity. An upper bound is derived for the error incurred at the detector output by the use of a finite number of terms in the optimum detector expansion. Error probabilities are calculated for the case in which the detector outputs may be assumed approximately normally distributed. From an output "signal-to-noise" ratio point of view, it is shown that the performance of the optimum detector and approximate detectors will differ negligibly if the above eigenvalues have squared magnitudes much less than unity. Some upper bounds are derived for the largest eigenvalue (magnitude).

I. Introduction

THE problem of detecting normal signals in normal additive noise has been studied by Price [1]–[4], Middleton [5]–[7], Turin [8], [9], and Kailath [10], [11]. Price's [1]–[3], Turin's, and Kailath's work pursue the question of scatter-path communications and are concerned with an optimum probability computing receiver for N'ary signals. While Middleton studies only the binary situation, he considers [5], [6] a general decision theory formulation of the problem of detecting the presence or absence of a normally distributed process in background normal noise. More recently [7], Middleton has been concerned with the somewhat more general problem of discriminating between two Gaussian processes.

While the results of the above analyses are of great interest, they are generally not in a form which lends itself readily to application. The reason for this is that in the case of discrete sampling, one must invert matrices of high order, while in the case of continuous observation, one must find the solution to certain integral equations for which the method of solution is usually not known. The purpose of the present paper is threefold:

1) To present, in the case of continuous observation, a derivation of the optimum detector structure for discriminating between two Gaussian processes, termed the "inverse operator" approach, which is an alternate approach to Middleton's [7].
2) To obtain, via the Neumann series expansion and the inverse operator approach, approximations to the optimum detector (valid in certain practically meaningful situations) which require the inversion of at most one linear operator (and sometimes none).
3) To examine some of the approximate detector structures in 2).

The derivation is carried through for complex-valued Gaussian processes, thereby making the results applicable to a wide class of nonstationary Gaussian narrow-band processes (and, of course, to narrow-band stationary processes in general).

II. The Inverse Operator Approach

The statistical problem we are concerned with is the following: Given a finite record of a complex-valued normally distributed process $Z(t)$, find an optimum test to determine whether this process has covariance $K(t, s)$ or $L(t, s)$ (assuming it is known *a priori* that only these two covariances are possible). Because of the continuum of values present in $Z(t)$, it is necessary in forming probability density functions to deal, at least initially, with some representation which involves only a finite number of random variables. Two representations have been used in the past:

1) Representation by discrete samples of $Z(t)$; see Middleton [5], [7] and Priec [1]–[3].
2) Representation by finite orthogonal series; see Grenander [12] and Davenport and Root [13].

The method of approach is to solve the statistical problem with the finite variable approximation, and then allow the approximation to become arbitrarily fine. Thus, in the discrete sample approximation, the number of samples in the finite record of $Z(t)$ would be allowed to increase indefinitely, while in the orthogonal representation of $Z(t)$ arbitrarily high-order terms would be included in the expansion of $Z(t)$. We will use the latter method of representation to derive the inverse operator approach.

It is presumed that the reader is familiar with the Karhunen-Loéve orthonormal series expansion of a random process.[1] Briefly, if $Z(t)$ has a continuous covariance function $K(t, s)$ defined as

$$K(t, s) = \overline{Z^*(t)Z(s)} = K^*(s, t), \tag{1}$$

* Received by the PGIT, September 20, 1960.
† Alcom Inc., Cambridge, Mass. Formerly Appl. Res. Lab., Sylvania Electronic Systems, Waltham, Mass.

[1] See [13], p. 96.

hen it may be represented by the series

$$Z(t) = \sum_{n=1}^{\infty} \zeta_n \phi(t), \tag{2}$$

here the ζ_n are uncorrelated random variables given by

$$\zeta_n = \int_T Z(t)\phi_n^*(t)\, dt \quad \text{where} \quad \int_T |\phi_n(t)|^2\, dt = 1 \tag{3}$$

nd $\phi_n(t)$ is the nth eigenfunction of the homogeneous redholm integral equation,

$$\int_T K(s, t)\phi_n(s)\, ds = \lambda_n \phi_n(t) \tag{4}$$

n which λ_n is the nth eigenvalue and $\phi_i(t)$ is orthogonal o $\phi_k(t)$ for $j \neq k$. The subscript T denotes an interval f integration of duration T. It should be noted that nce we assume that $Z(t)$ is normally distributed, ζ_n s also normally distributed. In this case, ζ_i is not only ncorrelated, but also independent of ζ_k for $j \neq k$. It is eadily found by direct evaluation from (3) and use of 4) that

$$\overline{|\zeta_n|^2} = \lambda_n > 0; \tag{5}$$

e., that the variance of the nth coefficient is just the nth igenvalue of (4). The joint probability density function f the first N coefficients is

$$W_\zeta(\zeta_1, \zeta_2, \cdots, \zeta_N) = \left(\frac{1}{\pi}\right)^N \frac{1}{\prod_1^N \lambda_n} \exp\left[-\sum_1^N \frac{|\zeta_n|^2}{\lambda_n}\right]. \tag{6}$$

If the covariance function of $Z(t)$ were $L(t, s)$, then an ntirely analogous development would follow where now ve have eigenfunctions $\psi_n(t)$, eigenvalues μ_n, and coefficients η_n. The joint probability density function of the first N coefficients would then be

$$W_\eta(\eta_1, \eta_2, \cdots, \eta_N) = \left(\frac{1}{\pi}\right)^N \frac{1}{\prod_1^N \mu_n} \exp\left[-\sum_1^N \frac{|\eta_n|^2}{\mu_n}\right]. \tag{7}$$

For the subsequent development, it will be necessary o represent the summation in the argument of the exponents in (6) and (7) in an integral form,

$$\begin{aligned} \sum_1^N \frac{\zeta_n^* \zeta_n}{\lambda_n} &= \sum_1^N \frac{\zeta_n^*}{\lambda_n} \int_T Z(t)\phi_n^*(t)\, dt = \int_T Z(t) Y_N^*(t)\, dt \\ \sum_1^N \frac{\eta_n^* \eta_n}{\mu_n} &= \int_T Z(t) X_N^*(t)\, dt \end{aligned} \tag{8}$$

here

$$\begin{aligned} Y_N(s) &= \sum_1^N \frac{\zeta_n \phi_n(s)}{\lambda_n} \\ X_N(s) &= \sum_1^N \frac{\eta_n \psi_n(s)}{\mu_n}. \end{aligned} \tag{9}$$

If both sides of the first equation in (9) are multiplied by $K(s, t)$ and integrated with respect to s over the interval T, and if both sides of the second equation are multiplied by $L(s, t)$ and integrated in the same way, it is found that Y_N and X_N are solutions of the following Fredholm integral equations:

$$\begin{aligned} \int_T K(s, t) Y_N(s)\, ds &= Z_N^\phi(t); \qquad t \in T \\ \int_T L(x, t) X_N(s)\, ds &= Y_N^\psi(t); \qquad t \in T \end{aligned} \tag{10}$$

in which

$$\begin{aligned} Z_N^\phi(t) &= \sum_1^N \zeta_n \phi_n(t) \\ Z_N^\psi(t) &= \sum_1^N \eta_n \psi_n(t); \end{aligned} \tag{11}$$

i.e., $Z_N^\phi(t)$ is the first N terms in the orthonormal expansion of $Z(t)$ with basis functions $\phi_n(t)$, while $Z_N^\psi(t)$ is the first N terms in the orthonormal expansion of $Z(t)$ with basis functions $\psi_n(t)$. Eqs. (10) can be written in linear operator form as

$$\begin{aligned} KY_N &= Z_N^\phi \\ LX_N &= Z_N^\psi, \end{aligned} \tag{12}$$

where the K and L operators are readily identified in (10). When the inverse operators exist, they are linear, and provide solutions for Y_N and X_N by the formal operations[2]

$$\begin{aligned} Y_N &= K^{-1} Z_N^\phi \\ X_N &= L^{-1} Z_N^\psi. \end{aligned} \tag{13}$$

In general, the inverse of an integral operator (such as K and L are, usually) is not an integral operator. However, if one permits the use of delta functions and their derivatives, one may frequently be able to express the operations K^{-1} and L^{-1} in the formalism of integral operators. In such cases, one may write

$$\begin{aligned} Y_N(t) &= \int_T K^{-1}(s, t) Z_N^\phi(s)\, ds; \qquad t \in T \\ X_N(t) &= \int_T L^{-1}(s, t) Z_N^\psi(s)\, ds; \qquad t \in T \end{aligned} \tag{14}$$

where $K^{-1}(s, t)$ and $L^{-1}(s, t)$ are kernels which may contain delta functions and derivatives of delta functions. Formally, then, we may rewrite (6) and (7) as

[2] By K^{-1} we mean the linear operator which provides a solution to the equation $Ky = z$, where y and z are suitably well-behaved functions.

$$W_\zeta(\zeta_1, \zeta_2, \cdots, \zeta_N) = \left(\frac{1}{\pi}\right)^N \frac{1}{\prod_1^N \lambda_n} \cdot \exp\left[-\int_T \int_T Z(t) Z_N^{*\phi}(s) K^{-1}(t, s)\, dt\, ds\right] \quad (15)$$

$$W_\eta(\eta_1, \eta_2, \cdots, \eta_N) = \left(\frac{1}{\pi}\right)^N \frac{1}{\prod_1^N \mu_n} \cdot \exp\left[-\int_T \int_T Z(t) Z_N^{*\psi}(s) L^{-1}(t, s)\, dt\, ds\right]$$

where the readily deduced symmetry properties

$$\begin{aligned} K^{-1}(s, t) &= (K^{-1}(t, s))^* \\ L^{-1}(s, t) &= (L^{-1}(t, s))^* \end{aligned} \quad (16)$$

have been used.

As is well known, an optimum rule for categorizing a set of samples as coming from one of two possible populations is the Neyman Person Rule. This involves a computation of the ratio of the two possible density functions (the likelihood function) when the actual sample values are inserted into the argument of the density functions, and a comparison of this ratio with respect to a threshold. Other optimum rules are known [5]. However, all perform the same operation, although with different thresholds. In the case of normal density functions, it is more convenient (and just as valid) to compare the logarithm of the likelihood ratio with a threshold. Examination of (15) indicates that if N coefficients in the orthonormal expansion of $Z(t)$ are used, the optimum test involves a computation of

$$Q_N = \int_T \int_T Z(t)\{Z_N^{*\phi}(s) K^{-1}(t, s) - Z_N^{*\psi}(s) L^{-1}(t, s)\}\, dt\, ds \quad (17)$$

followed by a comparison with a threshold. Now we desire to let N increase without bound so as to have a complete characterization of the observed record. If the Singular case[3] is not to occur (where it is possible to achieve a perfect test with a finite record of data), then Q_N must remain finite as $N \to \infty$. Moreover, note that

$$\begin{aligned} \lim_{N\to\infty} Z_N^{\phi}(s) &= Z(s) \\ \lim_{N\to\infty} Z_N^{\psi}(s) &= Z(s). \end{aligned} \quad (18)$$

Thus in the Regular (or Nonsingular) case, the optimum detector involves the computation of

$$Q = \int_T \int_T Z(t) Z^*(s)\{K^{-1}(t, s) - L^{-1}(t, s)\}\, dt\, ds, \quad (19)$$

which is the desired result of this section.

[3] The most recent discussion of the Singular case may be found in [14].

It should be noted that although the derivation has involved the use of eigenfunctions and eigenvalues, the final test (evaluation of Q) involves only a determination of the difference of the inverse kernels followed by a double integration involving the observed data.

One might be tempted to examine (14) and take the limit as $N \to \infty$, arriving at

$$\begin{aligned} Y(t) &= \int_T K^{-1}(s, t) Z(s)\, ds; \qquad t \in T \\ X(t) &= \int_T L^{-1}(s, t) Z(s)\, ds; \qquad t \in T. \end{aligned} \quad (20)$$

However, this may not be permissible since, as will subsequently be shown, both $Y(t)$ and $X(t)$ are *of necessity* unbounded if the test is to be Regular. What is permissible is to define the difference

$$W(t) = \lim_{\Lambda\to\infty} [Y_N(t) - X_N(t)] = \int_T Z(s)\{K^{-1}(s, t) - L^{-1}(s, t)\}\, ds \quad (21)$$

in terms of which

$$Q = \int_T Z(t) W^*(t)\, dt < \infty. \quad (22)$$

The unbounded nature of $X(t)$ and $Y(t)$ is deduced from the fact that the integrals

$$\begin{aligned} \int_T Z(t) Y^*(t)\, dt &= \infty \\ \int_T Z(t) X^*(t)\, dt &= \infty \end{aligned} \quad (23)$$

with probability 1 when $Z(t)$ is a complex-valued normally distributed random process with covariance $K(t, s)$ or $L(t, s)$ and (22) is satisfied. To prove these integrals unbounded, assume that $Z(t)$ has covariance $K(t, s)$. From (8) we have

$$\lim_{N\to\infty} \sum_1^N \frac{|\zeta_n|^2}{\lambda_n} = \lim_{N\to\infty} \int_T Z(t) Y_N^*(t)\, dt = \int_T Z(t) Y^*(t)\, dt \quad (24)$$

when the limit exists. Using (5), we find that the mean value of the sum in (24) equals N. It follows that the infinite sum of mean values does not converge. But, according to Kolmogorov [15], a necessary condition for the convergence of a sum of independent random variables is that the sum of the means converge. Thus we conclude that the first integral in (23) is unbounded. The proof that the second integral in (23) also must be unbounded results trivially from (22). An entirely analogous argument leads to the same results when it is assumed that the covariance of the observed process is $L(t, s)$.

In view of the above, it is likely that although the inverse operators K^{-1} and L^{-1} may individually behave

uite radically, their difference will exhibit a much moother behavior for a Regular test.

III. Expansion of the Optimum Detector

The inversion of K or L is a difficult task, in general. Thus it appears advisable to consider techniques which lleviate the difficulties of this inversion process. In this ection we consider expansions of the optimum detector n a series of quadratic forms which, when applicable, nvolve the inversion of at most one linear operator. It is lear that such an approach will be practically useful at he present time only if the linear operator to be inverted s in the class of operators for which techniques of in- ersion are available. In the discussion to follow, it will e seen that the operator to be inverted is one of the ollowing: K, L, $K + L$, or N (where N is the operator hose kernel is the covariance of an assumed additive oise).

A. Neumann Series Expansion

Approximately optimum detector structures may be ound when it is possible to express the operation of the ptimum detector in a rapidly convergent series expansion f suboptimum operations. In this case, the first term or o will suffice to represent the actual optimum detector tructure. The series expansions of the optimum detector onsidered here depend upon the use of a linear operator xpansion of the type

$$[M + I]^{-1} = I - M + M^2 - M^3 \cdots = \sum_{n=0}^{\infty} (-1)^n M^n \qquad (25)$$

hich is called a Neumann Series expansion. It is well nown[4] that a sufficient condition for the convergence[5] f this expansion is that

$$|| M ||^2 = \operatorname*{Max}_x \frac{(Mx, Mx)}{(x, x)} < 1 \qquad (26)$$

here the scalar product definition

$$(f, g) = \int_T f(t) q^*(t)\, dt \qquad (27)$$

as been used and $|| M ||$ is defined as the norm of the ransformation M. Since[6]

$$(Mx, Mx) = (M^*Mx, x), \qquad (28)$$

it follows that

$$|| M ||^2 = \operatorname*{Max}_x \frac{(M^*Mx, x)}{(x, x)}. \qquad (29)$$

It is well known[7] that for a symmetric[8] matrix A, stationary values of the ratio $(Ax, x)/(x, x)$ are eigenvalues of

$$Ax = \lambda x. \qquad (30)$$

Thus

$$\operatorname*{Max}_x \left| \frac{(Ax, x)}{(x, x)} \right| = \operatorname{Max} \{ | \lambda_A | \} \qquad (31)$$

where $\{ | \lambda_A | \}$ is the set of absolute values of the eigenvalues of (30).

Since M^*M is both symmetric and positive,[9] a sufficient condition for the convergence of the Neumann series (25) is that

$$\operatorname{Max} \{\lambda_{M^*M}\} < 1. \qquad (32)$$

When M is symmetric, $M^*M = M^2$, and $\operatorname{Max} \{\lambda_{M^*M}\} = \operatorname{Max} \{\lambda_M^2\}$. In this case, the inequality in (32) may be replaced by

$$\operatorname{Max} \{ | \lambda_M | \} < 1. \qquad (33)$$

B. Detector Expansions

One type of expansion of the optimum detector to be discussed depends upon the operators K and L being "close" to one another and requires the inversion of either K or L. Let us suppose that the inverse of K, K^{-1}, may be determined. Let the difference operator D be defined as

$$D = L - K. \qquad (34)$$

Then, providing the inverse of K exists,[10] we may express L^{-1} in the form

[4] For example, see [16], ch. IV, sect. 67. This expansion is also alled the Liouville-Neumann expansion. The operator M^s is equal o the operator M applied s times in succession, and $M^0 \equiv I$, the dentity operator.

[5] The convergence implied is that the integrated (over T) quared magnitude of the difference between $[M + I]^{-1}x$ and $\sum_{n=0}^{N} (-1)^n M^n x$ approaches zero as $N \to \infty$ (limit in the mean efinition).

[6] The adjoint operator M^* may be defined as that operator hich satisfies

$$(M^*x, x) = (x, Mx).$$

M were an integral operator with kernel $M(t, s)$, then the adjoint ernel would be $M^*(t, s)$ where the asterisk here has the usual omplex conjugate interpretation.

[7] See [16], ch. VI, sect. 93, theorem on p. 232. This theorem was pointed out to the author by Dr. T. Kailath, M.I.T. Res. Lab. of Electronics.

[8] A symmetric operator M is one which equals its adjoint M^*, *i.e.*, $M = M^*$.

[9] A linear operator A is positive if $(Ax, x) \geq 0$ and positive definite if $(Ax, x) > 0$ for $x \neq 0$. In general, the operators K and L are only positive and symmetric. When the original noise processes to be discriminated contain additive white noise, K and L become positive definite. In fact, if the spectral intensity of this white noise is N_0, then (Kx, x) and (Lx, x) are $\geq N_0(x, x) > 0$.

[10] One may rigorously justify the factorization shown in (35) when K^{-1} is a bounded linear operator, *i.e.*, when it has a finite norm. Now (*cf.* [16], sect. 104), K^{-1} will be bounded if and only if K is positive definite. Thus the validity of the factorizations in (35) may be open to question if K is merely positive. A possible way to proceed in the case that K is only positive is to add ϵI to the operators K and L, where ϵ is a small positive quantity. In this way the new operators $K' = K + \epsilon I$ and $L' = L + \epsilon I$ become positive definite. The ϵ may be carried through in the subsequent expansions and allowed to approach 0 at the end. Of course, one must establish that the conditions required for the convergence of the series expansions of the optimum detector are not violated as $\epsilon \to 0$. Actually, for Regular tests (the only ones considered here), it is doubtful that one will need to let $\epsilon \to 0$, since the addition of ϵI to K and L is equivalent to adding white noise of spectral intensity ϵ to our original processes. If the test is Regular to start with, the addition of a small amount of white noise cannot make the test Singular, and, on the basis of physical reasoning, cannot change the error probabilities of the optimum detector significantly.

$$L^{-1} = [K + D]^{-1} = [(I + DK^{-1})K]^{-1}$$
$$= K^{-1}[I + DK^{-1}]^{-1}. \quad (35)$$

According to Section III-A, if

$$|| DK^{-1} || < 1,$$

we may form the expansion

$$[I + DK^{-1}]^{-1} = I - DK^{-1} + (DK^{-1})^2 \cdots$$
$$= I + \sum_{j=1}^{\infty} (-DK^{-1})^j \quad (36)$$

which leads to

$$K^{-1} - L^{-1} = K^{-1}[DK^{-1} - (DK^{-1})^2 \cdots]$$
$$= K^{-1} \sum_{j=1}^{\infty} (-1)^{j+1}(DK^{-1})^j. \quad (37)$$

The operation of the optimum detector is thus expressible as a sum of quadratic form as follows:

$$Q = ([K^{-1} - L^{-1}]z, z)$$
$$= (K^{-1} DK^{-1}z, z) - (K^{-1} DK^{-1} DK^{-1}z, z) \cdots$$
$$= \sum_{j=1}^{\infty} (-1)^{j+1}(K^{-1}[DK^{-1}]^j z, z) = \sum_{j=1}^{\infty} (-1)^{j+1} Q_j \quad (38)$$

where the quadratic form Q_j is given by

$$Q_j = (K^{-1}[DK^{-1}]^j z, z). \quad (39)$$

If L is invertible, an analogous development yields

$$Q = \sum_{j=1}^{\infty} F_j; \qquad || DL^{-1} || < 1 \quad (40)$$

where

$$F_j = (L^{-1}[DL^{-1}]^j z, z). \quad (41)$$

An expansion involving the inversion of the sum operator

$$S = K + L \quad (42)$$

is made by noting that

$$K^{-1} - L^{-1} = 2(S - D)^{-1} - 2(S + D)^{-1}$$
$$= 2S\{(I - DS^{-1})^{-1} - (I + DS^{-1})^{-1}\} \quad (43)$$

and using the appropriate Neumann expansions. Because of the symmetry in the expression in (43), we have the expansion

$$K^{-1} - L^{-1} = 4S^{-1} \sum_{j=1}^{\infty} (DS^{-1})^{2j-1}; \quad || DS^{-1} || < 1 \quad (44)$$

in which only odd powers of DS^{-1} are present. In this case, the optimum detector has the expansion (neglecting the irrelevant factor of 4)

$$Q = \sum_{j=1}^{\infty} V_j; \qquad j \text{ odd} \quad (45)$$

where

$$V_j = (S^{-1}[DS^{-1}]^j z, z). \quad (46)$$

When additive noise is present in the two process to be discriminated, we may express K and L as

$$L = A + N$$
$$K = B + N. \quad (47)$$

Here, N is the linear integral operator whose kernel is the covariance of the additive noise, and A, B are the linear integral operators corresponding to the covariance of the processes that it is desired to discriminate between. If N^{-1} may be obtained, we may expand L^{-1} and K^{-1} as follows:

$$L^{-1} = N^{-1}[I + AN^{-1}]^{-1}$$
$$= N^{-1}\left[I + \sum_{j=1}^{\infty} (AN^{-1})^j(-1)^j\right]$$
$$K^{-1} = N^{-1}[I + BN^{-1}]^{-1} \quad (48)$$
$$= N^{-1}\left[I + \sum_{j=1}^{\infty} (BN^{-1})^j(-1)^j\right]$$

provided

$$|| AN^{-1} || < 1$$
$$|| BN^{-1} || < 1. \quad (49)$$

In this case, the optimum detector has an expansion of the form

$$Q = \sum_{j=1}^{\infty} [Q_{A_j} - Q_{B_j}](-1)^{j+1} \quad (50)$$

where

$$Q_{A_j} = (N^{-1}[AN^{-1}]^j z, z)$$
$$Q_{B_j} = (N^{-1}[BN^{-1}]^j z, z). \quad (51)$$

When it is desired to discriminate between the presence and absence of a process in additive noise, one of the pair A, B equals zero. Assuming $B = 0$, it follows that $Q_{B_j} = 0$, all j, and

$$Q = \sum_{j=1}^{\infty} (-1)^{j+1}(N^{-1}[AN^{-1}]^j z, z) = \sum_{j=1}^{\infty} Q_{A_j};$$
$$|| AN^{-1} || < 1. \quad (52)$$

When the additive noise is white

$$N = N_0 I; \qquad N^{-1} = \frac{1}{N_0} I \quad (53)$$

and use of (50) or (52) then requires no operator inversions.

IV. Conditions for Rapid Convergence of Detector Expansion

In this section we will be concerned with establishing ufficient conditions for the rapid convergence of the equence of quadratic forms involved in the optimum letector expansion of the previous section. Consider irst the expansion (38). We may express the ratio of an dd term (term with odd subscript) to the previous even erm (term with even subscript) as

$$\frac{Q_{2j+1}}{Q_{2j}} = \frac{(K^{-1}[DK^{-1}]^{2j+1}z, z)}{(K^{-1}[DK^{-1}]^{2j}z, z)}$$

$$= \frac{(K^{-1}(DK^{-1})^j\, D(K^{-1}\, D)^j K^{-1}z, z)}{(K^{-1}(DK^{-1})^j(DK^{-1})^j z, z)}. \tag{54}$$

f we define the operator

$$P_j = K^{-1}(DK^{-1})^j = (K^{-1}\, D)^j K^{-1} = P_j^*, \tag{55}$$

hen the ratio in (54) may be expressed as

$$\frac{Q_{2j+1}}{Q_{2j}} = \frac{(P_j DP_j z, z)}{(P_j K P_j z, z)} = \frac{(DP_j z, P_j z)}{(KP_j z, P_j z)} = \frac{(Dy_j, y_j)}{(Ky_j, y_j)} \tag{56}$$

where the function y_j is defined as

$$y_j = P_j z. \tag{57}$$

By a simple application of the calculus of variations, ne may determine that stationary values of the ratio[11]

$$\frac{(Dy, y)}{(Ky, y)} \tag{58}$$

where D, K are symmetric, are equal to the eigenvalues f[12]

$$Dy = \lambda Ky. \tag{59}$$

Since D and K are symmetric, the eigenvalues are real. f K is positive (as it is here), the maximum value of the nagnitude of the ratio in (58) will be a stationary value nd thus equal to the absolute value of the eigenvalue of 59) with the largest magnitude. Note, however, that we nay assume this absolute value is finite only if K is ositive definite [since then (Ky, y) has a positive greatest ower bound]. Since K is invertible (by hypothesis), we ave an equivalent eigenvalue problem[13]

$$K^{-1}\, Dy = \lambda y \tag{60}$$

where $K^{-1}D$ is a nonsymmetric operator. If we denote the eigenvalue with the largest magnitude as ρ, then

$$| Q_{2j+1} | \leq | \rho | Q_{2j}. \tag{61}$$

Note that Q_{2j} is non-negative, while Q_{2j+1} can be positive or negative.[14] It follows from (61) and the fact that equality may be obtained in (61) when y_j is the eigenfunction of (59) corresponding to the eigenvalue ρ, that $| \rho | < 1$ is both a necessary and sufficient condition for convergence of the detector expansion (38). This latter statement presumes that Q_1 and Q_2 are finite. Noting that Q_{2k} is positive, we deduce from (6) that

$$Q_{2j+s} = \epsilon_s | \rho |^s Q_{2j} \tag{62}$$

where $-1 \leq \epsilon_s \leq 1$ for s odd and $0 \leq \epsilon_s \leq 1$ for s even. It then follows that (38) may be expressed as

$$Q = \sum_{k=1}^{2j} (-1)^{k+1} Q_k - \epsilon Q_{2j} \tag{63}$$

where the error coefficient ϵ is bounded by

$$-\frac{| \rho |}{1 - | \rho |^2} < \epsilon < \frac{| \rho |}{1 - | \rho |}. \tag{64}$$

One may conclude that rapid convergence will be assured if

$$| \rho | \ll 1 \tag{65}$$

and this being the case, the magnitude of the error in using the first two terms is bounded by

$$Q_2 \frac{| \rho |}{1 - | \rho |} \approx | \rho | Q_2. \tag{66}$$

It should be noted that (61) yields an upper bound on a term Q_p relative to a preceding *even*-order term.[15] Thus, as far as the present development is concerned, we must include at least the first two terms, in a discussion of error bounds.

Exactly analogous statements may be made with regard to the expansion (40) where the appropriate eigenvalue equation is

$$Dz = \lambda Lz \quad \text{or} \quad L^{-1}\, Dz = \lambda z \tag{67}$$

and the symbol F replaces Q on the right side of (63) and (66).

[11] In the analogous matrix problem, we have Courant's Theorem (*cf.* p. 56 of [17]).

[12] For some discussion of the analogous matrix eigenvalue quation, see [18], p. 74. See also problems (78), (80), (83), and 85) on pp. 115 and 116.

[13] As already mentioned,[10] when K is positive, but not positive efinite, K^{-1} will be an unbounded linear operator. This does not revent $K^{-1}D$ from being a bounded linear operator. Strictly peaking, one may rigorously justify the equivalence of the eigen- alue problems (59) and (60) only if K is positive definite. When K is positive, we may invoke the ϵ argument[10] to justify our manip- lations.

[14] A quadratic form G_j with a generic expression

$$G_j = (T_1[T_2T_1]^j z, z)$$

in which T_1, T_2 are symmetric may be expressed as

$$G_j = \begin{cases} (T_2 f_j, f_j); & j \text{ odd} \\ (T_1 h_j, h_j); & j \text{ even} \end{cases}$$

where $f_j = T_1[T_2T_1]_2^{j-1/2} z$ and $h_j = [T_2T_1]^{j/2} z$. Thus, the term G_j will be non-negative for odd j, if T_2 is positive and non-negative for even j, if T_1 is positive.

[15] This restriction could be removed if D were positive definite. Then one would be able to show that $Q_{j+s} \leq | \rho |^s Q_j$ for j odd or even.

The situation is somewhat different in the expansion in (45) where we may express

$$Q = \sum_{\substack{k=1 \\ k \text{ odd}}}^{2j-1} V_j + \epsilon V_{2j} \tag{68}$$

in which

$$| \epsilon | < \frac{| \rho |}{1 - | \rho |^2}. \tag{69}$$

In (69), ρ is the eigenvalue of

$$Dz = \lambda Sz \quad \text{or} \quad S^{-1} Dz = \lambda z \tag{70}$$

with the largest magnitude. Since even-order terms are absent in the expansion (45), we may obtain $| \epsilon | V_2 \approx | \rho | V_2$ for $| \rho | \ll 1$ as a bound on the magnitude of the error caused by the use of only the first term.

The expansions of (50) and (52) differ from the previous ones in that the operators A and B in the quadratic forms Q_{A_j} and Q_{B_j} (51) are positive, while the operator D in Q_j, F_j, and V_j is not. As a result, Q_{A_j} and Q_{B_j} are non-negative for all j. Moreover, it may be shown by arguments similar to those leading up to (61) that

$$Q_{C_{j+s}} < \rho_C^s Q_{C_j}; \qquad C = A, B \tag{71}$$

where ρ_C is the largest eigenvalue of

$$Cz = \lambda Nz \quad \text{or} \quad N^{-1}C = \lambda z. \tag{72}$$

Since A, B, and N are positive, all eigenvalues of (72) are positive. In view of (71), the expansion (50) may be expressed as

$$Q = \sum_{k=1}^{j} [Q_{A_k} - Q_{B_k}](-1)^{k+1} + [\epsilon_A Q_{A_j} - \epsilon_B Q_{B_j}](-1)^{j+2} \tag{73}$$

where

$$-\frac{\rho_C^2}{1 - \rho_C^2} \leq \epsilon_C \leq \frac{\rho_C}{1 - \rho_C^2}; \qquad C = A, B. \tag{74}$$

It is clear that rapid convergence of (73) will occur when ρ_A and ρ_B are individually much less than unity (assuming of course that Q_{A_1} and Q_{B_1} are bounded).

V. Error Probabilities for Large Sample Size

Strictly speaking, a suboptimum detector may be said to adequately represent an optimum detector when the error probabilities of the associated hypothesis testing problem satisfactorily approximate those of the optimum detector. Thus a rigorous justification for using the first n terms of a series expansion of the optimum detector would be a demonstration that the resulting error probabilities satisfactorily approximate those of the optimum detector. Such a demonstration appears to be quite difficult except in the case where the observation interval T is sufficiently large. For this latter case, the detector output probability distributions may be expected to be approximately normally distributed.[16] In this section, we will assume the normal assumption to be sufficiently accurate. When the first term or so in the previous detector expansions does accurately approximate the optimum detector, one may expect the normal approximation to be valid if low error probabilities are to be obtained. This expectation follows from the fact that a rapidly convergent detector expansion implies that the two processes to be discriminated are "close" to one another, and thus a long observation time is required to discriminate between them.

On the assumption that a quadratic form is normally distributed, only its mean and variance need be computed to determine its probability density function. In the discussion to follow, we will need the following two statistical averages[17]

$$\begin{aligned} \overline{(Mz, z)} &= \operatorname{Tr} [MR_z] \\ \overline{(Mz, z)(Nz, z)} &= \operatorname{Tr} [MR_z] \operatorname{Tr} [NR_z] + \operatorname{Tr} [MR_zNR_z] \end{aligned} \tag{75}$$

where the overline denotes an ensemble average, R is an integral operator whose kernel is the covariance of the z process, M, N are symmetrical operators, and

$$\operatorname{Tr} [P] = \int_T P(t, t)\, dt \tag{76}$$

is the trace of the integral operator whose kernel is $P(t, s)$.

In order to keep the following discussion as uncluttered as possible, we will compare the performance of the optimum and suboptimum detector when the threshold levels of the respective detectors are individually adjusted such that for each test the two types of error probabilities are equal. (This would correspond to binary symmetric operation in a communication system.) Then one may readily determine that this common error probability for a test (assuming normal statistics prior to thresholding) is given by

$$\alpha = \phi\left[\frac{| m_1 - m_2 |}{\sigma_1 + \sigma_2}\right] \tag{77}$$

where $m_{1,2}$ and $\sigma_{1,2}^2$ denote the means and variances, respectively, of the detector output (prior to thresholding) for the two possible hypotheses, and

$$\phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx. \tag{78}$$

In the following discussion we will consider only the expansion (38). The other expansions presented in Section III may be handled in an analogous fashion. Let $Q_K^{(n)}$,

[16] See [6], sect. 17.2-1, p. 234, and [19], sect. (3), p. 326.

[17] For a complex normally distributed process (with zero mean), one may demonstrate that $\overline{z(t_1)z^*(t_2)z(t_3)z^*(t_4)} = \overline{z(t_1)z^*(t_2)} \cdot \overline{z(t_3)z^*(t_4)} + \overline{z(t_1)z^*(t_4)} \cdot \overline{z(t_3)z^*(t_2)}$. Note that for a complex normal process with zero mean $\overline{z(t_a)z(t_b)} = 0$. See [20], p. 72.

$Q_L^{(n)}$ denote the sum of the first n terms in (38) for the ypotheses that the input process has covariance $K(t, s)$ r $L(t, s)$, respectively. When $n = \infty$, we arrive at the ptimum detector outputs for the two hypotheses. These ill be denoted by Q_K and Q_L. With the aid of (75), one ay readily determine that

$$\overline{Q_K^{(n)}} = \sum_{j=1}^{n} (-1)^{j+1} \operatorname{Tr}[P^j];$$

$$\overline{Q_L^{(n)}} = \operatorname{Tr}[P] + (-1)^{n+1} \operatorname{Tr}[P^{n+1}]$$

$$\operatorname{ar} Q_K^{(n)} = \sum_{j=2}^{2n} (j-1)(-1)^j \operatorname{Tr}[P^j]; \tag{79}$$

$$\operatorname{Var} Q_L^{(n)} = \operatorname{Tr}[P^2] + 2n \operatorname{Tr}[P^{2n+1}] + (2n-1) \operatorname{Tr}[P^{2n+2}]$$

here $P = K^{-1}D$ and Var X denotes the variance of X. se of (79) in (77) allows the determination of the error robability α_n associated with a suboptimum detector hich uses the first n terms of (38). We will make a etailed comparison between α, the error probability of ne optimum detector, and α_1, the error probability of ne suboptimum detector which uses only the first term f (38), *i.e.*, $n = 1$. From (77) and (79), these error robabilities are[18]

$$_1 = \phi\left[\frac{\operatorname{Tr}[P^2]}{\sqrt{\operatorname{Tr}[P^2]} + \sqrt{\operatorname{Tr}[P^2] + 2\operatorname{Tr}[P^3] + \operatorname{Tr}[P^4]}}\right];$$

$$\alpha = \phi\left[\frac{\sum_{j=2}^{\infty} (-1)^j \operatorname{Tr}[P^j]}{\operatorname{Tr}[P^2] + \sqrt{\sum_{j=2}^{\infty} (j-1)(-1)^j \operatorname{Tr}[P^j]}}\right]. \tag{80}$$

Now

$$\operatorname{Tr}[P^n] = \sum_{j=1}^{\infty} \lambda_j^n \tag{81}$$

here $\{\lambda_j\}$ is the set of eigenvalues of P. From (81) it is eadily shown that

$$\frac{\operatorname{Tr}[P^{2+s}]}{\operatorname{Tr}[P]} = \epsilon_s \mid \rho \mid^s \tag{82}$$

here $0 \leq \epsilon_s \leq 1$ for s even and $-1 \leq \epsilon_s \leq 1$ for s odd, nd ρ is the eigenvalue of $P = K^{-1}D$ that has the largest agnitude. By factoring Tr $[P^2]$ from the numerator nd $\sqrt{\operatorname{Tr}[P^2]}$ from the denominator of (80), and then sing (82) followed by appropriate Taylor series expan- ons, one finds that

$$\alpha \approx \phi[\tfrac{1}{2}\sqrt{\operatorname{Tr}[P^2]}\ \{1 - \tfrac{1}{2}\epsilon_1 \mid \rho \mid + \tfrac{1}{4}\epsilon_2 \mid \rho \mid^2\}]$$

$$_1 \approx \phi[\tfrac{1}{2}\sqrt{\operatorname{Tr}[P^2]}\ \{1 - \tfrac{1}{2}\epsilon_1 \mid \rho \mid - \tfrac{1}{4}(\epsilon_2 + \epsilon_1^2) \mid \rho \mid^2\}] \tag{83}$$

[18] Both $\lim_{n\to\infty} \operatorname{Tr}[P^n]$ and $\lim_{n\to\infty} n\operatorname{Tr}[P^n]$ must be zero from onsiderations of convergence of the series for Q_K and Var Q_K. ctually, as may be seen from (81) and (82), the vanishing of r$[P^n]$ and nTr$[P^n]$ is a consequence of the assumptions that Tr$[P^2]$ finite and that the absolute values of the eigenvalues of P are ss than 1.

where we have included terms of no higher order than $\mid \rho \mid^2$ in the ϕ arguments. Further Taylor series expansions[19] yield

$$\left| \frac{\alpha - \alpha_1}{\alpha} \right| < \tfrac{3}{4} \mid \rho \mid^2 (\tfrac{1}{4} \operatorname{Tr}[P^2] + 1). \tag{84}$$

The comparison of two tests on the basis of error probabilities is, for small error probabilities, a very conservative type of comparison. When error probabilities are small, large percentage changes in error probabilities can frequently be caused by very small changes in observation time, SNR, or other parameters. For communication or radar applications, where SNR's are readily changed, it would appear that a better comparison of two tests would be on the basis of a suitably defined SNR. In our problem, a suitable definition of SNR appears to be the argument of the error function in (77), *i.e.*, the difference between the means under the two hypotheses divided by the sum of the standard deviations. For Gaussian detector outputs (prior to thresholding), such a definition results in the optimum detector (error-probability-wise) having the maximum SNR.

Examination of (83) shows that η, the percentage difference in SNR between the optimum detector and suboptimum detector [consisting of the first term in (38)], satisfies the inequality[20]

$$\eta < \tfrac{3}{4} \mid \rho \mid^2. \tag{85}$$

Thus, from an SNR point of view, the condition $\mid \rho \mid \ll 1$ is sufficient to justify the use of only the first term in (38).

Presuming that $\mid \rho \mid \ll 1$, we see from (83) that a low error probability requires Tr $[P^2]$ to be much larger than unity. The inequalities $\mid \rho \mid \ll 1$ and Tr $[P^2] \gg 1$ are not incompatible. From (81), we see that satisfaction of these two inequalities implies that the sequence $\{\lambda_j^2\}$ must be a very slowly decreasing sequence (where $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$).

VI. Some Upper Bounds on $\mid \rho \mid$

We present here some bounds on $\mid \rho \mid$ that may be of some use. A straightforward application of the calculus of variations shows that

$$\mid \rho \mid^2 = \operatorname*{Max}_x \left| \frac{(Px, x)}{(x, x)} \right|^2 \tag{86}$$

[19] Note that

$$\phi(A + \delta A) = \phi(A) - \delta A \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{A^2}{2}\right]\left\{1 - \frac{\delta A^2}{2} - \frac{\delta^2 A^2}{6}(\xi^2 - 1)\right\}$$

where $A < \xi < A + \delta A$, and that

$$\mid \dot{\phi}(A)/\phi(A) \mid < A\left[1 + \frac{1}{A^2}\right].$$

[20] Again ignoring terms of order higher than ρ^2.

where ρ is the eigenvalue of (the possibly nonsymmetric operator) P with the largest magnitude.[21] An application of the Schwartz Inequality shows that

$$\left| \frac{(Px, x)}{(x, x)} \right|^2 \leq \frac{(Px, Px)}{(x, x)} = \frac{(P^*Px, x)}{(x, x)}. \tag{87}$$

Thus

$$| \rho | \leq || P || = \sqrt{\operatorname{Max}_x \frac{(Px, Px)}{(x, x)}}. \tag{88}$$

For an operator M with kernel $M(t, s)$

$$\begin{aligned}(Mx, x) = \int_T \int_T M(t, s)x(t)x^*(s)\, dt\, ds &\leq \int_T \int_T | M(t, s) | \, | x(t) | \, | x(s) | \, dt\, ds \\ &= \int_T \int_T \sqrt{| M(t, s) |} \, | x(t) | \, \sqrt{| M(t, s) |} \, | x(s) | \, dt\, ds \\ &\leq \sqrt{\int_T \int_T | M(t, s) | \, | x^2(t) | \, dt\, ds \int_T \int_T | M(t, s) | \, | x^2(s) | \, dt\, ds}\end{aligned} \tag{89}$$

where the last inequality results from an application of the Schwartz Inequality. Let

$$C_M = \operatorname{Max}_s \left\{ \int_T | M(t, s) | \, ds \right\}. \tag{90}$$

Then we see that

$$\begin{aligned}\int_T \int_T | M(t, s) | \, | x^2(t) | \, dt\, ds &\leq C_M \int_T | x^2(t) | \, dt \\ \int_T \int_T | M(s, t) | \, | x^2(s) | \, dt\, ds &\leq C_{M^*} \int_T | x^2(t) | \, dt.\end{aligned} \tag{91}$$

Using (95) in (93), it follows that

$$\frac{(Mx, x)}{(x, x)} \leq \sqrt{C_M C_{M^*}}. \tag{92}$$

If we let $M = P^*P = M^*$, then

$$\begin{aligned}|| P ||^2 &\leq C_{P^*P} \\ &= \operatorname{Max}_s \left\{ \int_T \left| \int_T P(t, x)P^*(s, x)\, dx \right| dt \right\} \\ &\leq \operatorname{Max}_x \left\{ \int | P(t, x) | \, dt \right\} \operatorname{Max}_s \left\{ \int | P(s, x) | \, dx \right\} \\ &= C_P C_{P^*}.\end{aligned} \tag{93}$$

Thus we have the series of inequalities[22]

$$| \rho | \leq || P || \leq \sqrt{C_{P^*P}} \leq \sqrt{C_P C_{P^*}}. \tag{94}$$

In the case that the kernel $P(t, s)$ has the symmetric form

$$P(t, s) = \sum_{k=1}^{N} f_k(t) f_k^*(s) r_k(s - t) \tag{95}$$

it is shown in the Appendix that

$$| \rho |^2 \leq \sum_{k=1}^{N} \operatorname{Max}_{t \in T} | f_k^2(t) | \operatorname{Max}_f | R_k(f) | \tag{96}$$

where $R_k(f)$ is the Fourier transform of $r_k(\tau)$,

$$R_k(f) = \int_{-\infty}^{\infty} r_k(\tau) e^{-i2\pi f\tau}\, d\tau. \tag{97}$$

It is also shown in the Appendix that for $N = 1$, $f_k(t) = 1$ and $R_k(f)$ positive, the upper bound given by (96) approaches arbitrarily close to $| \rho |$ as $T \to \infty$.[23]

When the two processes to be discriminated contain an additive white noise component of spectral intensity N_0, we may obtain a useful upper bound on $| \rho |^2$ as follows:

$$\begin{aligned}| \rho | \leq || P || &= || DK^{-1} || \leq || D || \, || K^{-1} || \\ &= \frac{1}{N_0} || A - B ||.\end{aligned} \tag{98}$$

The second inequality is valid if D and K^{-1} are bounded linear operators.[24] The last equality (47) follows from the fact that $|| K^{-1} || = 1/N_0$.[25]

From (98) we deduce that a sufficient condition for using the first term in the expansion of the optimum detector is that the norm (or the largest eigenvalue, since they are identical for symmetric operators) of the difference operator $A - B$ be much smaller than the spectral intensity of the additive white noise. If the processes whose covariances are the kernels of the operators A, B are stationary, then from (96) we deduce that another (but less tight) sufficient condition is that the

[21] This result in (86) is more general than (31), where the operator is assumed symmetric.

[22] Inequalities of these types for matrix operators can be found in [17], pp. 66 and 67, and [22]. The derivations leading to (94) are a generalization of a derivation shown to the author by Dr. R. Price, M.I.T., Lincoln Lab. Price derived the inequality $| \rho | < C_P$ for a symmetric operator, P.

[23] According to Kailath, this fact has been demonstrated by Szego [21].

[24] See [16], p. 149.

[25] It is assumed that A and B are integral operators, or, more to the point, that the eigenvalues of A and B have zero as a limit point.

aximum value of the difference spectrum be much smaller an N_0 Application of (94) yields

$$\frac{1}{N_0} \| A - B \| \leq \frac{1}{N_0} \operatorname{Max}_t \int_T | A(t, s) - B(t, s) | \, ds. \quad (99)$$

hus another sufficient condition for the use of the ptimum detector expansion is that

$$\frac{1}{N_0} \operatorname{Max}_t \int_T | A(t, s) - B(t, s) | \, ds \ll 1. \quad (100)$$

Appendix

It is sufficient to demonstrate (96) for $N = 1$. We have

$$| \rho | = \operatorname{Max}_{z(t)} \left| \frac{\int_T \int_T z(t) z^*(s) f(t) f^*(s) r(s - t) \, dt \, ds}{\int_T | z(t) |^2 \, dt} \right|. \quad (101)$$

et $z_T(t)$ be a truncated version of $z(t)$ defined by

$$z_T(t) = \begin{cases} 0; & t \notin T \\ z(t); & t \in T. \end{cases} \quad (102)$$

hen, by an obvious change in variable, the double ntegral in (101) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_T(t) z_T^*(s) f(t) f^*(s) r(s - t) \, dt \, ds = \int_{-\infty}^{\infty} k_T^*(\tau) r(\tau) \, d\tau \quad (103)$$

here $k_T(\tau)$ is the autocorrelation function of $z_T(t) f(t)$,

$$k_T(\tau) = \int_{-\infty}^{\infty} z_T^*(t) f^*(t) z_T(t + \tau) f(t + \tau) \, dt. \quad (104)$$

By Parseval's Theorem,

$$\int_{-\infty}^{\infty} k_T^*(\tau) r(\tau) \, d\tau = \int_{-\infty}^{\infty} K_T^*(f) R(f) \, df \quad (105)$$

here $K_T(f)$, $R(f)$ are the Fourier transforms of $k_T(\tau)$ nd $r(\tau)$ respectively. Noting that $K_T(f)$ is positive, e find

$$\int_{-\infty}^{\infty} K_T^*(f) R(f) \, df \leq \int_{-\infty}^{\infty} K_T^*(f) | R(f) | \, df \leq \operatorname{Max}_f | R(f) | \int_{-\infty}^{\infty} K_T(f) \, df. \quad (106)$$

ut

$$\int_{-\infty}^{\infty} K_T(f) \, df = k_T(0) = \int_{-\infty}^{\infty} | z_T(t) |^2 | f(t) |^2 \, dt \leq \operatorname{Max}_{t \in T} | f(t) |^2 \int_{-\infty}^{\infty} | z_T(t) |^2 \, dt. \quad (107)$$

Use of inequalities (106) and (107) show that

$$| \rho | \leq \operatorname{Max}_{t \in T} | f(t) |^2 \operatorname{Max}_f | R(f) |. \quad (108)$$

When $f(t)$ is unity, it may be readily seen from the above that $| \rho |$ may be expressed in the form

$$| \rho | = \operatorname{Max}_{W(f)} \left| \int_{-\infty}^{\infty} W(f) R(f) \, df \right| \leq \operatorname{Max}_f | R(f) | \quad (109)$$

where $W(f)$ is a positive function with unit area whose Fourier transform is nonzero only over an interval of duration T. As $T \to \infty$, $W(f)$ can be made to approach arbitrarily close to the unit impulse. Note that the bound on the right side of (109) is actually attained when $W(f)$ is an impulse located at that value of f for which $| R(f) |$ is maximum. Thus as $T \to \infty$, $| \rho | \to \operatorname{Max}_f | R(f) |$.

Acknowledgment

The author would like to acknowledge with pleasure the benefit of several helpful discussions with Dr. A. Wouk, Applied Research Laboratory, Sylvania.

References

[1] R. Price, "Optimum detection of random signals in noise with applications to scatter-multipath communication, part I," IRE Trans. on Information Theory, vol. IT-2, pp. 125–135; December, 1956.
[2] ——, "Notes on Ideal Receivers for Scatter Multipath," M.I.T. Lincoln Lab., Lexington, Mass., Group Rept. 34–39; May 12, 1955.
[3] ——, "Error Probabilities for the Ideal Detection of Signals Perturbed by Scatter and Noise," M.I.T. Lincoln Lab., Lexington, Mass., Group Rept. 34–40; October 3, 1955.
[4] —— and P. E. Green, "Signal Processing in Radar Astronomy—Communication via Fluctuating Multipath Media," M.I.T. Lincoln Lab., Lexington, Mass., Rept. No. 234; 1960.
[5] D. Middleton, "On the detection of stochastic signals in additive normal noise—part I," IRE Trans. on Information Theory, vol. IT-3, pp. 86–121; June, 1957.
[6] ——, "Introduction to Statistical Communication Theory," McGraw-Hill Book Co., Inc., New York, N. Y., p. 925; 1960.
[7] ——, "On the detection of stochastic signals in additive normal noise—part II," IRE Trans. on Information Theory, vol. IT-6, pp. 349–360; June, 1960.
[8] G. Turin, "Communication through noisy, random-multipath channels," 1956 IRE Convention Record, pt. 4, pp. 154–156.
[9] ——, "Error probabilities for binary symmetric ideal reception through nonselective slow fading and noise," Proc. IRE, vol. 46, pp. 1603–1619; September, 1958.
[10] T. Kailath, "Correlation detection of signals perturbed by a random channel," IRE Trans. on Information Theory, vol. IT-6, pp. 361–366; June, 1960.
[11] ——, "Optimum receivers for randomly varying channels," in "Proceedings of the Fourth London Symposium on Information Theory," Butterworths Scientific Press, London, Eng.; 1961.
[12] U. Grenander, "Stochastic processes and statistical inference," *Arkiv. Math.*, vol. 1, pp. 195–277; 1950.
[13] W. B. Davenport, Jr., and W. L. Root, "An Introduction to the Theory of Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y.; 1958.
[14] D. Middleton, "On singular and nonsingular optimum (Bayes) tests for the detection of normal stochastic signals in normal noise," IRE Trans. on Information Theory, vol. IT-7, pp. 105–113; April, 1961.
[15] A. Kolmogorov, "Foundations of Probability Theory," Chelsea Publishing Co., New York, N. Y., p. 67; 1956.
[16] F. Riesz and B. Nagy, "Functional Analysis," Frederick Ungar Publishing Co., New York. N. Y.; 1955.
[17] E. Bodewig, "Matrix Calculus," North-Hall, Amsterdam; 1956.
[18] F. B. Hildebrand, "Methods of Applied Mathematics," Prentice-Hall, Inc., Englewood Cliffs, N. J.; 1952.
[19] C. W. Helstrom, "Statistical Theory of Signal Detection," Pergamon Press, New York, N. Y.; 1960.
[20] J. L. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y.; 1953.
[21] G. Szego, "Bertrage Zur Theorie der Toeplitzschen Formen," *Math. Z.*; 1920.
[22] A. Brauer, "Limits for the characteristic roots of matrices," *Duke Math. J.*, vol. 13, pp. 387–395; 1946.

Error Correcting Codes and Their Implementation for Data Transmission Systems*

J. E. MEGGITT†

Summary—Presented here is a practical automatic error-correcting system that may be applied to many data transmission problems. It is particularly suited to the correction of bursts of errors and so may be applied to the problem of the transmission of data over telephone networks.

The attractive feature of the system is its remarkable simplicity from the point of view of implementation. It is so simple that it can readily be incorporated into much existing equipment.

In the system, messages are transmitted in blocks and each block is coded separately. The codes used within the blocks are cyclic codes. This means that coders and decoders employ linear feedback shift registers to form check digits and to correct errors.

The basic ideas are presented in terms of the hardware components to which the system gives rise and analyzed afterwards in terms of mathematics so that it is easy for the engineer to see, at once, what is involved.

The theory usually applies to binary messages in which data is transmitted serially. However, an extension is included which shows how the same ideas may be applied to binary codes in which information is sent in parallel.

Introduction

THE problem of automatic error correction is a central one in data transmission theory. Many existing communication systems are inherently noisy, and one has the simple choice between an expensive improvement to them and the adoption of an error-correcting system if one wishes to transmit data accurately. It is becoming more and more clear that computers need to communicate with each other over long distances and that data for them must also be communicated, so the problem is one that must be faced.

The existing telephone system already provides an excellent world-wide communication system, so it is highly desirable to utilize it for these new purposes. However, the system is inherently noisy, largely due to impulse noise generated in exchanges, so that any automatic error-correcting system which enables it to be employed is of great value. In the new data transmission fields, the aim is also to transmit information as rapidly as possible, because the more information that can be sent per second along a given line, the cheaper the transmission. Thus, it is even conceivable that it may be worth increasing the speed of a reliable system until errors start to occur, and this is a sensible thing to do provided, of course, that these errors may be automatically corrected. Fortunately, there exists a very simple type of error correcting system which is cheap to implement and which is particularly suitable for correcting bursts of errors; this will be described. There is still a fair amount of freedom within the system so that the code adopted can be matched to the kind of noise expected. Tests are presently in progress to determine the noise that different telephone lines produce, so that shortly it should prove possible to make an exact recommendation about what code to use in any given situation.

These remarks have been directed towards telephone transmission systems. However, the same error correcting systems find an application in radio transmission and in the magnetic tape and disk recording of data. In the latter case, data may be much more densely packed if errors that are caused by small imperfections in the magnetic materials can be corrected automatically.

Contents

The next section will describe the general apparatus that is required. The following one will give examples showing how it is employed, and the last will show some extensions of the theory.

Codes

The data transmitted is going to be supposed binary, so that messages consist of zeros and ones. Further, in the first part it is supposed that it is transmitted serially, one digit at a time. In the last section of this paper, there are some extensions for the case where the data is sent several digits at a time.

The coding procedure consists of splitting the message into blocks and adding to the information digits in each block certain redundant digits which are functions of the information digits. The redundancy is such that even though errors occur, there is enough information left for the message to be corrected. The encoding problem is to form these redundant digits simply, while the decoding problem is to reconstruct the correct message.

The check digits are made linear functions of the information digits; *i.e.*, the check digits are just the parities of certain groups of information digits. This is not necessary, of course, but it is certainly customary. The problem of code design that is left, having adopted this general strategy, is to choose the groups of digits for the parity checks intelligently, so that the coding process is simple and so that the sorts of error that are expected to occur are corrected.

* Received by the PGIT, December 2, 1960.
† IBM British Labs., Winchester, Hants, England.

Cyclic Codes

The codes that are considered here are a further restricted class of parity codes called cyclic codes. These are interesting because they are extremely simple to implement and because, historically, they were found to have many interesting properties. Abramson [1], Melas [2], and Fire [3], for example, realized that they had these properties without realizing their inherent simplicity.

Cyclic codes may be defined mathematically or they may be defined for the engineer by the implementation to which they give rise. The basis of this implementation is the feedback linear shift register and the cyclic properties of this give rise to the name "cyclic code." In this paper, the engineer's approach will be adopted, and the apparatus will first be described. Mathematics will only be introduced to analyze its behavior.

General Encoder for Cyclic Code

When a message block is transmitted, it is arranged to transmit the information digits first, and to follow them by the check digits. Let the total number of digits be n, and the number of check digits k.

The encoder is shown in Fig. 1 and consists mainly of a feedback shift register of length k. The c's denote connections to the register that can be made or not. The choice is left to the designer. It is convenient to describe the situation by putting $c = 1$ for a connection that is made and $c = 0$ for one that is not. $c_0 = 1$, because otherwise a shorter shift register would suffice.

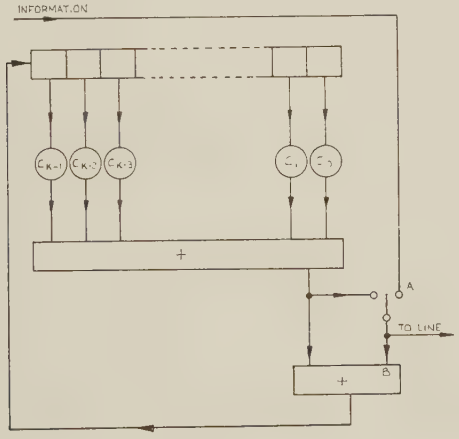

Fig. 1—Encoder.

The adders form the modulo two sum of their inputs and are really just EXCLUSIVE OR gates. The values that are given to the c's determine the properties of the codes, and when examples are considered, specific connections will be shown for specific codes.

The operation of the encoder is that switch A is first set so that the $n - k$ digits of information are directly transmitted as they arrive and are also fed to the shifting register which initially is empty. When all the information has been sent, the switch A is reversed so that the encoder's input is isolated, while the output from the shift register is transmitted. In this way, the k check digits are sent. While this happens the input to the shift register is zero, since the adder B has two similar inputs and so its output is zero. Thus, at the end of the transmission the shift register is again empty.

The number k is the number of shift stages. n is such that if the shifting register initially contained $100 \cdots 0$, and if it were simply fed back, without the presence of the switch A, then it would contain $100 \cdots 0$, again after exactly n shifts and not before.

It is now necessary to analyze the code defined by this encoder. If the digits of the message are $a_1, a_2, \cdots a_n$ (a_1 first), then it is seen that $a_{n-k+1} \cdots a_n$ are defined in terms of $a_1 \cdots a_{n-k}$, and it is possible to write down this relationship. To do this, it is necessary to describe the operation of the shift register mathematically, and for this it is convenient to denote its contents at any instant by the vector $\mathbf{y}$ and its contents after a shift by $\mathbf{Ty}$, where $\mathbf{T}$ is the k by k matrix

$$\mathbf{T} = \begin{bmatrix} c_{k-1} & c_{k-2} & \cdots & c_1 & c_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1}$$

(The first element of the column vector $\mathbf{y}$ corresponds to the contents of the left-hand element of the shift register.)

With this notation, the successive contents of the shift register in the encoder may be described. Initially it contains zero. It next contains $a_1\mathbf{x}$, where $\mathbf{x}$ is the vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2}$$

When the second digit a_2 arrives, it contains

$$a_1\mathbf{Tx} + a_2\mathbf{x}.$$

When the third digit a_3 arrives, it contains

$$a_1\mathbf{T}^2\mathbf{x} + a_2\mathbf{Tx} + a_3\mathbf{x}, \quad \text{etc.},$$

and this continues until all the information digits have arrived. However, even when the check digits are being formed, the check digits $a_{n-k+1} \cdots a_n$ are fed back into the shift register in exactly the same way. Thus, ultimately it contains

$$a_1\mathbf{T}^{n-1}\mathbf{x} + a_2\mathbf{T}^{n-2}\mathbf{x} + \cdots + a_{n-1}\mathbf{Tx} + a_n\mathbf{x} \tag{3}$$

Now, n was chosen so that $\mathbf{T}^n\mathbf{x} = \mathbf{x}$. Further, zero was, by design, fed back into the shift register for the last k shifts, so that it ultimately contains zero.

Hence,

$$a_1\mathbf{T}^{-1}\mathbf{x} + a_2\mathbf{T}^{-2}\mathbf{x} + \cdots + a_{n-1}\mathbf{T}^{-(n-1)}\mathbf{x} + a_n\mathbf{T}^{-n}\mathbf{x} = 0 \quad (4)$$

and this is the mathematical definition of a cyclic code. This form is necessary in order to analyze the errors the code will correct. It is this set of linear equations that defines the check digits in terms of the others.

General Decoder for Cyclic Codes

The decoder for a cyclic code is shown in Fig. 2. It contains primarily two shift registers, one of length n where the message is stored until it can be corrected. Obviously, the message must be stored until the check digits arrive because, until they have arrived, it is not clear what has to be corrected. The other shift register is a feedback one of length k and has the same connections made to it as the one in the encoder has.

The decoder also contains a detector. Its function is to detect certain configurations in the lower shift register. If one of these is detected, a one is emitted; this inverts the digit that is currently leaving the main shift register and also adds a one to the adder of the lower shift register. The configurations that have to be detected depend on the errors the code is designed to correct, and what these have to be will be indicated shortly.

The operation of the decoder is as follows. As the message is received, it is stored in the main shift register, while simultaneously the arriving digits are fed to the lower shift register. During this time, the detector is disconnected. It is convenient to denote the incoming digits by $a_1'a_2' \cdots a_n'$ which, of course, may differ from $a_1 \cdots a_n$.

The lower shift register is initially made to contain zero, so that when the first digit a_1' arrives it contains, exactly, as in the case of the encoder, $a_1'\mathbf{x}$. It next contains $a_1'\mathbf{T}\mathbf{x} + a_2'\mathbf{x}$, etc., so that when the complete message has arrived, it contains

$$a_1'\mathbf{T}^{-1}\mathbf{x} + a_2'\mathbf{T}^{-2}\mathbf{x} + \cdots + a_n'\mathbf{T}^{-n}\mathbf{x} = \mathbf{z}. \quad (5)$$

If there has been no error, so that the a''s are the same as the a's, then clearly $\mathbf{z} = 0$ while, in general, when there have been errors, $\mathbf{z} \neq 0$. By design, the $\mathbf{z}$'s will be distinct for the distinct errors that it is required to correct. At this stage, the apparatus can already be used for error detection. All errors that lead to nonzero $\mathbf{z}$'s will be detected.

However, when it is used for error correction, the procedure is that the detector is next switched on, while the input to the decoder is disconnected from the adder of the shifting register. This happens when $\mathbf{z}$ has been formed, and shifting is continued. As shifting continues, the contents of the lower register are operated on by $\mathbf{T}$, while digits of the uncorrected message leave the storage register. We choose the detector so that it will detect the pattern that occurs in the lower register whenever an erroneous digit has reached the right-hand end of the storage register. At the next shift, the detector emits a one which corrects the error, and at the same time a one is added to the feedback of the lower register to indicate that a somewhat simpler error pattern now remains to be corrected. The process continues until the entire message has left the main store.

It should be observed that no fresh message can be received while the present one is being corrected. Thus, if digits arrive continuously, some sort of tandem arrangement must be devised.

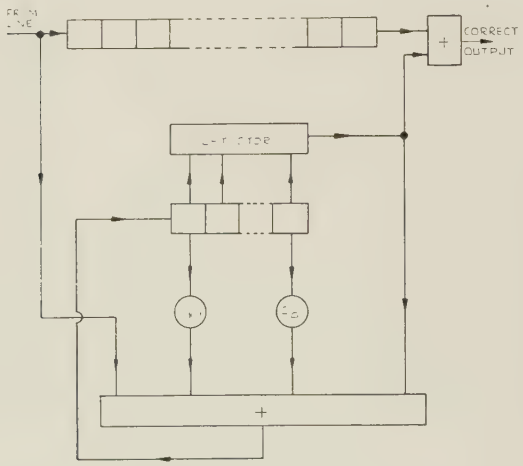

Fig. 2—Decoder.

The Detector

The states the detector has to detect depend on what the code is designed to do. The simplest detector is for the case where single error correction is required.

Single Error Detector

If there is, say, an error in the rth digit, then

$$a_r' = a_r + 1$$

while all the other a''s are equal to the corresponding a's. Hence, from (5)

$$\mathbf{z} = \mathbf{T}^{-r}\mathbf{x}. \quad (6)$$

Now the rth digit leaves the main storage register after $(r - 1)$ shifts. Thus, at this time the contents of the lower register are

$$\mathbf{T}^{r-1}(\mathbf{T}^{-r}\mathbf{x}) = \mathbf{T}^{-1}(\mathbf{x}).$$

Thus, it is necessary to detect the state $\mathbf{T}^{-1}\mathbf{x}$, and if this is done, any single error will be corrected.

$$\text{Since } \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and since } c_0 = 1,$$

$$\mathbf{T}^{-1}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

r any **T**. This just means that it is necessary to see when ie last element of the register contains one, while all the hers contain zero, and this specifies the detector. When the detector operates, it also emits a one into the dder of the shift register. It, therefore, effectively modifies s state **y**, say, to

$$\mathbf{y} + \mathbf{T}^{-1}\mathbf{x}.$$

In this case $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$, so that after the next shift, it ontains

$$\mathbf{T}(\mathbf{T}^{-1}\mathbf{x} + \mathbf{T}^{-1}\mathbf{x})$$
$$= 0.$$

The shift register now contains what it would if the essage were correct, and no further correction takes iace.

Double Adjacent Error Detector

If the code is designed to correct single and double djacent errors, then the detector must detect another tate besides $\mathbf{T}^{-1}\mathbf{x}$.

Suppose there are errors in the rth and $(r + 1)$th igits. Then in the same way as before

$$\mathbf{z} = \mathbf{T}^{-r}(1 + \mathbf{T}^{-1})\mathbf{x}. \tag{8}$$

The rth digit leaves the main storage register after $r - 1)$ shifts so that at this time, the lower register ontains

$$(\mathbf{T}^{-1} + \mathbf{T}^{-2})\mathbf{x}.$$

Hence, by arranging for the detector to detect $\mathbf{T}^{-1} + \mathbf{T}^{-2})\mathbf{x}$, the rth digit will be corrected.

Since, it has been arranged to add $\mathbf{T}^{-1}\mathbf{x}$ to the lower egister when the first correction takes place, the lower egister will contain after the next shift

$$\mathbf{T}[(\mathbf{T}^{-1} + \mathbf{T}^{-2})\mathbf{x} + \mathbf{T}^{-1}\mathbf{x}]$$
$$= \mathbf{T}^{-1}\mathbf{x}.$$

Now this is detected by the detector as for a single error; thus the $r + 1$th digit is also corrected and the register is left empty after the next shift.

The state $(\mathbf{T}^{-1} + \mathbf{T}^{-2})\mathbf{x}$ has a form which depends on **T** and so must be calculated from a knowledge of **T**. This is the form of the detector for a code that corrects single and double adjacent errors.

Detector for Burst Error Correction Errors Being Confined to Within a Length p $(p < k)$

The theory just presented extends in an obvious way to the case where bursts of errors up to those of length p have to be corrected. This means that when errors occur, they are spread over p or less consecutive digits, though not all p digits are necessarily wrong. It is assumed that the code used is such that this amount of correction is possible, and the problem is to see what the detector for this code should consist of.

The arguments of the last section may be followed exactly and it is found that for a burst code the detector must detect 2^{p-1} states, corresponding to the 2^{p-1} different burst patterns. These states are of the form

$$\mathbf{z} = \left(\mathbf{T}^{-1} + \sum_{i=2}^{p} q_i \mathbf{T}^{-i}\right)\mathbf{x} \tag{9}$$

where the q's take all combinations of values zero and one. Such a detector is quite easy to build, though it apparently gives rise to a certain complexity. However, there is an almost trivial arrangement that detects all states of the form (9). This is shown in Fig. 3, which shows primarily the basic feedback shift register of the decoder, but attached to it, the detector. This is designed to operate when the first $k - p$ digits of the shift register contain zero, while the last p digits are such as to cause the output from the feedback adder to be one. The assertion is that this arrangement detects all of the states of the form (9).

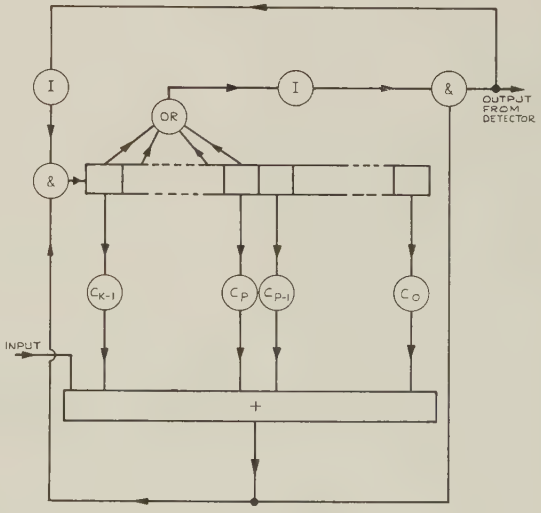

Fig. 3.

Proof

From the theory

$$\mathbf{T}^{-1}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{T}^{-2}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ t_1 \end{bmatrix}; \quad \mathbf{T}^{-3}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ t_1 \\ t_2 \end{bmatrix}, \quad \text{etc.,} \quad (10)$$

where operating with $\mathbf{T}$, it is seen that t_1, t_2, t_3 $\cdots$ are defined by

$$\begin{aligned} c_1 + t_1 &= 0 \\ c_2 + c_1 t_1 + t_2 &= 0 \\ c_3 + c_2 t_1 + c_1 t_2 + t_3 &= 0. \end{aligned} \quad (11)$$

Thus when the shift register contains $\mathbf{T}^{-1}\mathbf{x}$, the output from the adder is 1; when the shift register contains $\mathbf{T}^{-2}\mathbf{x}$, the output from the adder is $c_1 + t_1 = 0$; when the shift register contains $\mathbf{T}^{-3}\mathbf{x}$ the output from the adder is $c_2 + c_1 t_1 + t_2 = 0$, etc.

Hence, when the shift register contains $\mathbf{T}^{-1}\mathbf{x}$ and a linear combination of $\mathbf{T}^{-i}\mathbf{x}(i = 2 \cdots p)$, the output from the feedback adder is exactly one and, of course, the first $k - p$ elements of the shift register are zero. Now, when the first $k - p$ elements are zero, the output from the adder is one for 2^{p-1} states, and zero for 2^{p-1} states. Hence, the arrangement just described detects exactly those states of the form (9), and no more, and this completes the proof.

In the implementation previously described, the detector emits a one into the adder of the feedback shift register when it operates. In the new simplified arrangement, this may be done by having a separate two-input adder in the feedback loop for this purpose. However, it is simplest to observe that the effect of the addition is to feedback zero into the shifting register. Consequently, the same effect may be achieved by arranging for the operation of the detector, to disconnect the feedback loop, and this is shown in Fig. 3.

It should be carefully observed that this theory of the detector works so long as the code is such that burst correction up to length p is in principle possible. If one uses for example, a detector for $p = 2$ on a code that is only designed for single error correction, then what happens is that ambiguity occurs between single and double adjacent errors. Single errors in certain positions will be treated, and erroneously corrected, as double errors in other positions.

It should be noticed also that $p \ll k$, because, in fact, $p - 1$ digits of the k check digits described the nature of the error burst, while the other $k - p + 1$ have to be sufficient to position the burst.

Detector for General Cyclic Codes

A general cyclic code corrects various patterns of errors. If the code is such that an error pattern $1q_2q_3 \cdots$ can be corrected, then exactly as in (9) the detector in the decoder should be such that it detects the state

$$\mathbf{z} = (\mathbf{T}^{-1} + \sum_{i=2} q_i \mathbf{T}^{-i})\mathbf{x}. \quad (12)$$

It is also necessary that the code should correct, and the detector should detect, all simpler patterns, but presumably this will be required anyway. In general, there will not be the attractive simplicity shown in the last section, though some simplification is often possible.

Conclusion of Description of Apparatus

This concludes the description of the apparatus. The main problem that now needs to be discussed is the connections that need to be made to the shift registers to produce codes with assigned properties. The next section will contain some rules for the construction of some very powerful codes. However, a complete search through all possible sets of connections might lead to some other useful codes, and it may be worthwhile to use a computer to examine exhaustively the properties of all codes that can be generated using the apparatus.

Single Error Correcting Codes

The connections required to produce single error correcting codes will first be considered.

As has been observed, a single error in the rth digit leads to $\mathbf{z} = \mathbf{T}^{-r}\mathbf{x}$.

The code will, therefore, be capable of correcting a single error if all the vectors $\mathbf{T}^{-r}\mathbf{x}$ are different for different r, and this is indeed the case by construction. The most efficient code will be obtained when the shift register is connected so as to produce a maximal length cycle of length $2^k - 1$ in it. The cycle structure [4] of shift registers is generally discussed in terms of the characteristic equation that the matrix $\mathbf{T}$ satisfies, and from (1) it is seen that this equation is

$$\mathbf{T}^k + c_{k-1}\mathbf{T}^{k-1} + c_{k-2}T^{k-2} + \cdots + c_1\mathbf{T} + 1 = 0. \quad (13)$$

The characteristic equations that produce maximal length cycles are well documented and lists of them have been published. Thus, these may be used for these single error correcting codes.

Example

A code of message length 7 with 3 check digits is obtained by using the characteristic equation

$$\mathbf{T}^3 + \mathbf{T} + 1 = 0 \quad (14)$$

which is a very simple case taken from the list. The connections to the basic register are shown in Fig. 4.

From (4) it is found that the coding equations are explicitly

$$a_{5+i} + a_{3+i} + a_{2+i} + a_{1+i} = 0 \quad \text{for} \quad i = 0, 1, 2.$$

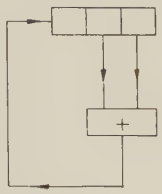

Fig. 4.

Double Adjacent Error Correcting Codes

The connections for these codes will next be considered. As has been observed in (8), it is required that the vectors

$$\mathbf{T}^{-r}\mathbf{x} \quad \text{and} \quad \mathbf{T}^{-s}(1 + \mathbf{T}^{-1})\mathbf{x}$$

all should be different for different values of r and s. This will be the case if the feedback shifting register is such that the vectors $\mathbf{x}$ and $(1 + \mathbf{T}^{-1})\mathbf{x}$ lie on different cycles of the same length.

The most efficient way of achieving this is to take a characteristic equation that has the form

$$(1 + \mathbf{T})M(k\mathbf{T}) = 0 \tag{15}$$

where $M(k\mathbf{T}) = 0$ is a characteristic equation that produces a maximum length cycle of length $2^k - 1$.

Proof

Consider the $k + 1$ by $k + 1$ matrix

$$\mathbf{T} = \left[\begin{array}{c|c} \mathbf{T}_1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{16}$$

where $\mathbf{T}_1$ is a matrix satisfying

$$M(k\mathbf{T}_1) = 0. \tag{17}$$

Then, clearly, the characteristic equation that $\mathbf{T}$ satisfies is just (15), though this newly-defined $\mathbf{T}$ has a different form. Now the cyclic structure is determined by the characteristic equation (except in exceptional circumstances) that $\mathbf{T}$ satisfies, so that this newly-defined $\mathbf{T}$ gives rise to a similar cycle structure. The cycle structure of this new $\mathbf{T}$ is now obvious.

There is a cycle of length $2^k - 1$ with vectors of the form

$$\left[\begin{array}{c} \mathbf{T}_1^{-r}\mathbf{x}_1 \\ \hline 0 \end{array}\right] \qquad r = 1 \cdots 2^k - 1$$

where $\mathbf{x}_1$ is the k by 1 column vector $\begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$

and there is another one of length $2^k - 1$ with vectors of the form

$$\left[\begin{array}{c} \mathbf{T}_1^{-r}\mathbf{x}_1 \\ \hline 1 \end{array}\right] \qquad r = 1 \cdots 2^k - 1.$$

$$\text{Now} \quad \left[\begin{array}{c} \mathbf{x}_1 \\ \hline 1 \end{array}\right] + \left[\begin{array}{c} \mathbf{T}_1^{-1}\mathbf{x}_1 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} (1 + \mathbf{T}_1^{-1})\mathbf{x}_1 \\ \hline 0 \end{array}\right]$$

so that the sum of two consecutive vectors on this cycle gives a vector on the other cycle, and this is exactly the the structure required.

Example

A code of message length 7 with 4 check digits can be obtained by taking

$$M(k\mathbf{T}) = \mathbf{T}^3 + \mathbf{T} + 1 \tag{18}$$

so that the code is described by the characteristic equation

$$\begin{aligned} (\mathbf{T} + 1)(\mathbf{T}^3 + \mathbf{T} + 1) &= 0 \\ \mathbf{T}^4 + \mathbf{T}^3 + \mathbf{T}^2 + 1 &= 0. \end{aligned} \tag{19}$$

This code is capable of correcting single and double adjacent errors. The basic shift register is shown in Fig. 5.

The coding equations are explicitly

$$a_{4+i} + a_{3+i} + a_{1+i} = 0 \qquad i = 0, 1, 2, 3.$$

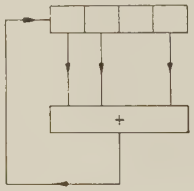

Fig. 5.

The detector in this decoder has to detect the states

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

This is equivalent to detecting two zeros in the first two shift elements and a one in the last, and this is in accordance with the rules for detectors to perform burst correction.

Fire Codes [3]

There exists an extension of the double adjacent error correction codes just described. These are derived from a characteristic equation

$$(\mathbf{T}^p + 1)M(k\mathbf{T}) = 0 \tag{20}$$

where p is prime to $2^k - 1$. This gives rise to codes of length $p(2^k - 1)$ with $p + k$ check digits. The chief virtue of this form is that the resulting cycle structure

is easy to analyze. For this purpose, it is best to consider the $p + k$ by $p + k$ matrix $\mathbf{T}$ defined by

$$\mathbf{T} = \left[\begin{array}{c|c} \mathbf{T}_1 & 0 \\ \hline 0 & \mathbf{T}_2 \end{array}\right] \tag{21}$$

where $\mathbf{T}_1$ satisfies $M(k\mathbf{T}_1) = 0$, and $\mathbf{T}_2$ is the p by p matrix

$$\mathbf{T}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{22}$$

As before, this $\mathbf{T}$ is different from the matrix usually considered, but it satisfies the same characteristic equation, and hence it has the same cycle structure.

The basic cycle of length $p(2^k - 1)$ may be taken as consisting of vectors

$$\left[\begin{array}{c} \mathbf{T}_1^{-r}\mathbf{x}^1 \\ \hline \mathbf{T}_2^{-r}\mathbf{x}_2 \end{array}\right]$$

where $\mathbf{x}_1$ is the k by 1 vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \cdot \end{bmatrix}$$

and $\mathbf{x}_2$ is the p by 1 vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}.$$

Then, the vector

$$(1 + \sum_{i=1} q_i\mathbf{T}^{-i})\left[\begin{array}{c} \mathbf{x}_1 \\ \hline \mathbf{x}_2 \end{array}\right],$$

which is a linear combination of vectors on this basic cycle, will be found to be

$$\left[\begin{array}{c} (1 + \sum_{i=1} q_i\mathbf{T}^{-i})\mathbf{x}_1 \\ \hline 1 \\ 0 \\ 0 \\ \vdots \\ q_3 \\ q_2 \\ q_1 \end{array}\right].$$

Further, all other members of the cycle to which this vector belongs will contain in the lower p positions just a cyclic permutation of the p values shown in this vector.

Thus, in general, the vectors

$$(1 + \sum_{i=1} q_i\mathbf{T}^{-i})\left[\begin{array}{c} \mathbf{x}_1 \\ \hline \mathbf{x}_2 \end{array}\right]$$

will be found to be on different cycles for different sets of q_i, each cycle being of length $p(2^k - 1)$, and this is exactly the structure required for a code that corrects sets of errors $1q_1q_2q_3 \cdots$.

The cycle will indeed be of length $p(2^k - 1)$, provided first that the q's are such that p cyclic shifts are required before the pattern shown repeats itself, and second that $(1 + \sum_{i=1} q_i\mathbf{T}_1^{-i})\mathbf{x}_1 \neq 0$. This latter condition may be ensured by taking a characteristic equation of sufficiently high degree for $\mathbf{T}_1$.

Example

When $p = 5$, it is found that errors 1, 11, 101, 111, 1111, and 1011 or 1101 may be corrected.

Thus, the characteristic equation

$$(\mathbf{T}^5 + 1)(\mathbf{T}^5 + \mathbf{T}^2 + 1) = 0$$
$$\mathbf{T}^{10} + \mathbf{T}^7 + \mathbf{T}^2 + 1 = 0 \tag{23}$$

gives a code of message length 155, containing 10 check digits, that is capable of correcting errors of the above form.

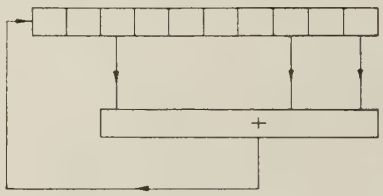

Fig. 6.

The shift connections are shown in Fig. 6, and the detector has to detect six patterns. These patterns may easily be calculated and are in fact

0 0 0 0 0 0 0 0 0 1 for error pattern 1

0 0 0 0 0 0 0 0 1 1 for error pattern 11

0 0 0 0 0 0 0 1 0 0 for error pattern 101

0 0 0 0 0 0 0 1 1 0 for error pattern 111

0 0 0 0 0 0 1 1 0 0 for error pattern 1111

and 0 0 0 0 0 0 1 0 0 1 for error pattern 1101

or 0 0 0 0 0 0 1 1 1 0 for error pattern 1011.

It is seen that the detector for the first four patterns can be exactly the detector for burst patterns of length 3. The other two patterns must be detected separately.

If this code were used in a practical situation, a good way of using it would be to use the simple detector for bursts of length 3, so that just bursts 1, 11, 101, and 111 are corrected. It will be remembered that after the entire correction has taken place, the contents of the feedback shift register in the decoder is zero. Hence, if the message has been sent correctly, or if errors 1, 11, 101, 111 have occurred and have been corrected, the shift register will be zero. Therefore, if at the end of this correction procedure, the register is not zero, this clearly indicates that some other error has occurred, and so this system may be used to detect the presence of a number of further errors.

General Burst Correcting Codes

It is clear that the Fire code [3] just described will correct all errors within a burst length $p + 1/2$, p odd; $p/2$, p even, as well as many more. This gives a straightforward way of constructing a general burst code.

For example, by taking

$$(\mathbf{T}^{11} + 1)(\mathbf{T}^6 + \mathbf{T}^4 + \mathbf{T}^3 + \mathbf{T} + 1) = 0, \tag{24}$$

it is possible to obtain a code of message length 693, 17 digits of which are check, and the code is capable of correcting all burst errors up to length 6, as well as many more.

It should be recalled that the apparatus required in the encoder and decoder for this is essentially a shift register of length 17, which is reasonable, while the decoder must also contain a shift buffer of length 693.

It should be observed that though this Fire code may seem inefficient for burst correction, in this example only about 3 more check digits are used than an optimum code would employ, and the difference seems immaterial.

As in the previous example, it is very simple to employ such a code as a burst correcting code, using the simplified detector in the decoder that has been described. The remaining redundancy that the code contains can then be used for the detection of other errors, and this is done by examining the shift register at the end of the correction procedure to see whether it is zero.

Bose-Chaudhuri Codes

Bose and Chaudhuri [5] have given a theory of cyclic codes for general multiple error correction. Their theory shows that if the characteristic equation is taken as

$$M(k\mathbf{T})N(\mathbf{T}) = 0 \tag{25}$$

where $N(\mathbf{S}) = 0$ is the characteristic equation that $\mathbf{S} = \mathbf{T}^3$ satisfies, and $\mathbf{T}$ satisfies $M(k\mathbf{T}) = 0$, then the resulting code is able to correct all double errors.

For example, if

$$M(k\mathbf{T}) = \mathbf{T}^4 + \mathbf{T}^3 + 1 \tag{26}$$

this implies

$$\mathbf{T}^{12} + \mathbf{T}^9 + \mathbf{T}^6 + \mathbf{T}^3 + 1 = 0.$$

Thus

$$\mathbf{S} = \mathbf{T}^3 \quad \text{satisfies}$$
$$\mathbf{S}^4 + \mathbf{S}^3 + \mathbf{S}^2 + \mathbf{S} + 1 = 0. \tag{27}$$

Consequently, the characteristic equation

$$(\mathbf{T}^4 + \mathbf{T}^3 + 1)(\mathbf{T}^4 + \mathbf{T}^3 + \mathbf{T}^2 + \mathbf{T} + 1) = 0$$
$$\mathbf{T}^8 + \mathbf{T}^4 + \mathbf{T}^2 + \mathbf{T} + 1 = 0 \tag{28}$$

according to the theory of Bose and Chaudhuri gives a code of message length 15, with 8 check digits, which is capable of correcting all double errors.

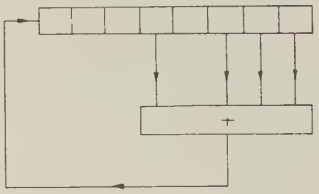

Fig. 7.

Fig. 7 shows the shift register connections. Explicitly, the code is

$$a_{8+i} + a_{4+i} + a_{2+i} + a_{1+i} = 0 \quad \text{for} \quad i = 0 \cdots 7.$$

The detector in the decoder must, of course, detect all 14 states of the form

$$\mathbf{T}^{-1}\mathbf{x} + \mathbf{T}^{-i}\mathbf{x}$$

for $i = 2 \cdots 15$.

The same theory shows that a triple error correcting code is achieved if the characteristic equation is taken as

$$M(k\mathbf{T})N(\mathbf{T})P(\mathbf{T}) = 0 \tag{29}$$

where $P(\mathbf{S}) = 0$ is the equation that $\mathbf{S} = \mathbf{T}^5$ satisfies.

Thus, in the previous example,

$$P(\mathbf{S}) = \mathbf{S}^2 + \mathbf{S} + 1 \tag{30}$$

and it is seen that in this way a triple error correcting code can be constructed with message length 15, 10 digits being check digits.

Unfortunately, the number of different patterns to be corrected is now very large ($\sim 15^2$), so the detector loses its simplicity. Other schemes then, in fact, become feasible, but to describe these is beyond the scope of this paper.

Expanded Double Error Correcting Errors

The previous theory shows how to construct double error correcting codes of message length $2^p - 1$, where there are $2p$ check digits or less. It is, in fact, possible to construct double error correcting codes of length $2^p + 1$ using $2p$ check digits, provided $2^p + 1$ is not divisible by 3.

The necessary results will merely be quoted. It may be shown that the characteristic equation

$$\mathbf{T}^2 + a\mathbf{T} + 1 = 0 \tag{31}$$

where a is a Galois field element belonging to $GF(2^p)$ gives a cycle structure which consists of $2^p - 1$ cycles each of length $2^p + 1$. This theory is like the previous theory, but vectors and matrices have elements belonging to $GF(2^p)$.

Further, if $2^p + 1$ is not divisible by 3, it may be proved that the vectors

$$\mathbf{x}, (1 + \mathbf{T}^{-i})\mathbf{x}$$

all lie on different cycles for $i = 1, 2, 3 \cdots 2^{p-1}$. Thus the cycle structure is suitable for building a double error correcting code.

$\mathbf{T}$ may be thought of as a 2 by 2 matrix with elements in $GF(2^p)$, or a matrix representation may be taken for the field elements, in which case $\mathbf{T}$ may be thought of as a $2p$ by $2p$ matrix with elements zero or one. This matrix will have the same cycle structure, so the characteristic equation is required for this $2p$ by $2p$ matrix. This may be obtained at once by using the equation that a satisfies to eliminate a from (31).

An example will perhaps clarify these points. Let a be defined by

$$a^4 + a + 1 = 0 \tag{32}$$

so that a belongs to $GF(2^4)$. Then, since from (31), $a = \mathbf{T}^{-1} + \mathbf{T}$, on substitution

$$(\mathbf{T}^{-1} + \mathbf{T})^4 + (\mathbf{T}^{-1} + \mathbf{T}) + 1 = 0.$$

Consequently

$$\mathbf{T}^8 + \mathbf{T}^5 + \mathbf{T}^4 + \mathbf{T}^3 + 1 = 0. \tag{33}$$

Thus, using the theory, a code having the shift connections shown in Fig. 8 will have 17 message digits, 8 of them check, and will be capable of correcting all double errors.

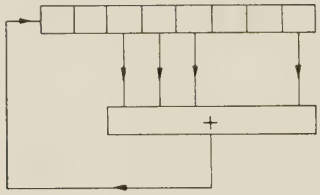

Fig. 8.

Explicitly, the coding equations will be

$$a_{10+i} + a_{7+i} + a_{6+i} + a_{5+i} + a_{4+i} + a_{1+i} = 0 \tag{34}$$

for $i = 0, 1, \cdots 7$.

This code is thus seen to be slightly superior in efficiency to the corresponding Bose-Chaudhuri [5] one.

Single Character Correcting Codes

Hitherto, all the theory has applied to binary messages, and as a consequence, modulo 2 arithmetic has been used to describe what happens. This is no hardship, of course, for messages usually are binary. However, there is a straight generalization of the theory in which the digits of the message take q values (q prime). The shift registers have elements with q stable states, and the characteristic equation describing the code has coefficients that belong to a prime field. This extension seems to have only academic interest, but there is a further similar extension in which the elements of the message take 2^p values. The shift registers have elements with 2^p stable states, and the characteristic equation has coefficients that belong to the Galois Field $GF(2^p)$. This extension is interesting because, when a binary message is transmitted p digits at a time, instead of serially, this is exactly the situation. Consequently, it is profitable to investigate whether any of these extended equations produce worthwhile codes, and also whether these complicated shift registers can be built out of multiple binary ones.

In the previous section, characteristic equations of this form were introduced, but only as an aid to analysis. Now it is being suggested that they should be used seriously.

Example

An example will show the possibilities. Consider the characteristic equation

$$\mathbf{T}^2 + b\mathbf{T} + 1 = 0 \tag{35}$$

where b belongs to $GF(4)$, and b is defined by

$$b^2 + b + 1 = 0$$

so that

$$b^3 = 1.$$

(If the theory of the last section is applied to this, it is found that a double error correcting code of message length 5, 4 of them check digits, is obtained. There is, thus, only one information digit, and the two coded messages it is possible to send are found to be 00000 and 11111. It is thus trivial that double error correction is possible.)

If, however, (35) is used in its own right, $\mathbf{T}$ is taken to have the form

$$\mathbf{T} = \begin{bmatrix} b & 1 \\ 1 & 0 \end{bmatrix} \tag{37}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and it is found that $\mathbf{T}^5\mathbf{x} = \mathbf{x}$.

Thus, the coding equations of the form (4) become

$$a_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ b \end{bmatrix} + a_3\begin{bmatrix} b \\ b \end{bmatrix} + a_4\begin{bmatrix} b \\ 1 \end{bmatrix} + a_5\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \tag{38}$$

and the message length is 5, while the number of check digits is 2. The a's of course take values 0, 1, b, b^2.

The basic shift register for this code is shown in Fig. 9. The shift elements take four values 0, 1, b, b^2, and the circle with b inside indicates multiplication by b.

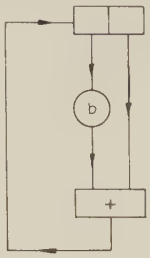

Fig. 9.

The cycle structure of the register clearly consists of 3 cycles each of length 5; one cycle starts with $\begin{bmatrix}1\\0\end{bmatrix}$, one with $\begin{bmatrix}b\\0\end{bmatrix}$, and one with $\begin{bmatrix}b^2\\0\end{bmatrix}$.

Consequently, the 15 vectors

$$b^i\mathbf{T}^r\mathbf{x}$$

are all different for different i and $r(i = 0, 1, 2; r = 0, 1 \cdots 4)$. Thus, the code defined by (38) is capable of single character error correction, where each character may be in error by 1, b or b^2, *i.e.*, in any possible way.

Encoder for Example

The encoder for this problem has exactly the form of the basic one shown in Fig. 1, but incorporates the register shown in Fig. 9.

Decoder for Example

The decoder has exactly the form of that shown in Fig. 2, except for the detector. It is seen that for an error of 1, the detector has to detect the state $\begin{bmatrix}0\\1\end{bmatrix}$, for an error b, the state $\begin{bmatrix}0\\b\end{bmatrix}$ for an error b^2, the state $\begin{bmatrix}0\\b^2\end{bmatrix}$.

Hence, the simplest arrangement is that shown in Fig. 10. The detector now has to determine not only whether to correct the output from the main register, but by how much.

The AND gate operates when the contents of the first element of the register is zero, and allows the contents of the second element to be added to the output from the main shift register.

Construction of Four-State Shift Register Elements

The construction of an encoder and decoder for this problem is now clear in principle, but it is not yet clear how to actually construct the basic shift register in Fig. 11.

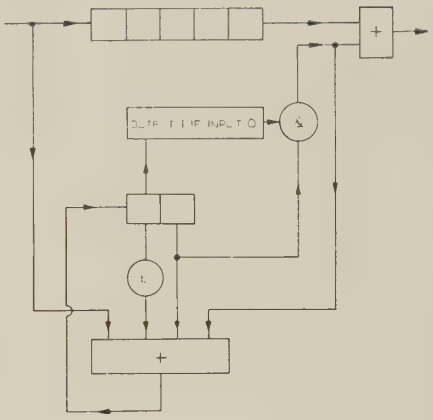

Fig. 10.

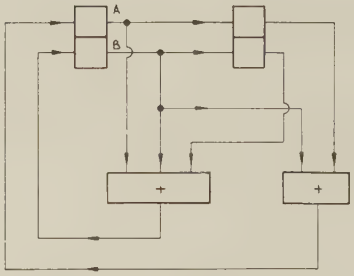

Fig. 11.

For this, a representation of the elements of the Galois Field is used. Any character of the message, or any element of the register, may be written

$$y = (A + Bb)$$

where A and B take values zero and one. y may be represented by the vector (A, B).

Thus, the state 0 is represented by (0, 0).

Thus, the state 1 is represented by (1, 0).

Thus, the state b is represented by (0, 1).

Thus, the state b is represented by (1, 1).

When two elements y_1 and y_2 are added together, as they are in the adder of the shift register, then clearly the A's and B's of the representation are added separately.

Further, when an element y is multiplied by b, it is found that

$$by = (Ab + Bb^2) \tag{39}$$

$$= B + (A + B)b$$

so that the vector representing by is $(B, A + B)$.

Thus, the shift register may be built from binary components that impose the correct relations on the A's and B's. The binary representation of the basic shift register is shown in Fig. 11.

General Single Character Correcting Codes

The ideas illustrated by this example generalize very easily. When the characters in the message take 2^p values, it is necessary first to find a characteristic equation of degree k with coefficient in $GF(2^p)$ that has a cycle structure of $2^p - 1$ cycles, each of length $h = 2^{pk} - 1/2^p - 1$, and this can, in general, be done. When this is done, the cycles can be written out and it will be found that either

the cycle that starts with $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ contains $\begin{bmatrix} b \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} b^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ etc., or

that it does not. If it does not, then clearly the result is a single character correcting code that corrects errors $1, b, b^2$, etc. If it does, then the theory breaks down, but clearly 2^{p-1} must be a factor of h. Hence, this is a general method of constructing single character correcting codes, provided 2^{p-1} is not a factor of h.

Examples

1) Characters taking 4 values; $b^2 + b + 1 = 0$. (40)

$\mathbf{T}^2 + b\mathbf{T} + 1 = 0$ gives a code length

5 with 2 check characters. (41)

$\mathbf{T}^4 + b^2\mathbf{T}^3 + b^2\mathbf{T}^2 + b\mathbf{T} + 1 = 0$ gives a code

length 85 with 4 check characters. (42)

Note that the above restriction prevents the finding of a code of length 21 with 3 check digits by this method.

2) Characters taking 8 values. The

field is defined by $c^3 + c + 1 = 0$. (43)

$\mathbf{T}^2 + c\mathbf{T} + 1 = 0$ gives a code length

9 with 2 check characters, (44)

$\mathbf{T}^3 + c\mathbf{T} + 1 = 0$ gives a code length

73 with 3 check characters. (45)

3) Characters taking 16 values. The

field is defined by $a^4 + a + 1 = 0$. (46)

$\mathbf{T}^2 + a\mathbf{T} + 1 = 0$ gives a code length

17 with 2 check characters. (47)

Thus, it is possible to construct in this way many codes for correcting single character errors in information that is transmitted in parallel. There is, of course, no reason why these ideas should not be applied to the construction of burst error correcting codes with messages whose characters take 2^p values. However, no suitable characteristic equations have yet been found. These ideas may perhaps be applied very profitably to magnetic tape recording where binary information is handled in parallel.

Conclusion

This completes the description of codes to be given. It is emphasized again that the apparatus needed for their implementation is as little as could reasonably be hoped for, and yet they have the power to turn an unusable communication system into a usable one.

The main problem for the future is to list the different shift register connections, together with the usable codes they produce, and to match these codes with the line characteristics that are found from measurements on communications systems.

Of the codes that have been described, those that promise most for practical situations seem to be the Fire codes, where bursts of errors up to length p are corrected, while the remaining potentialities of the code are used for additional error detection. In this case, there is the greatest simplicity, and additional protection is provided. Telephone engineers and others are understandably reluctant to employ a pure correcting code, because they are worried about the possible occurrence of catastrophic errors that might escape unnoticed, if some error detection were not included.

The Bose-Chaudhuri codes should be used in cases where errors occur randomly. It is fair to repeat, as has been pointed out, that the decoder for a double error correcting code is a little cumbersome, since its detector has to detect a fairly large number of patterns, while for a triple error correcting code, the number of patterns becomes very large.

It may well be that an optimum system contains a hierarchy of codes of the type described here, and that a message should be coded several times over at different levels. However, this is a problem for the future.

Acknowledgment

The author would like to acknowledge two interesting discussions, one with Prof. B. Elspas at Stanford University, and one with Prof. W. W. Peterson at M.I.T.

References

[1] N. M. Abramson, "A Class of Systematic Codes for Non-Independent Errors," Electronic Res. Lab., Stanford University, Stanford, Calif., Tech. Rept. No. 51; December 30, 1958.
[2] C. M. Melas, "A new group of codes for correction of dependent errors in data transmission," *IBM J. Res. & Dev.*, vol. 4, pp. 58–65; January, 1960.
[3] P. Fire, "A Class of Multiple-Error Correcting Binary Codes for Non-Independent Errors," Sylvania Electronics Systems, Rept. RSL-E-2, March, 1959; also presented at AIEE Mtg., Chicago, Ill., October, 1959.
[4] B. Elspas, "The theory of autonomous linear sequential networks," IRE Trans. on Circuit Theory, vol. CT-6, pp. 45–60; March, 1959.
[5] R. C. Bose and D. K. Ray-Chaudhuri, "On a class of error correcting binary group codes," *Information and Control*, vol. 3, pp. 68–79; March, 1960.
[6] J. E. Meggitt, "Error correcting codes for correcting bursts of errors," *IBM J. Res. & Dev.*, vol. 4, pp. 329–334; July, 1960.
[7] W. W. Peterson and D. T. Brown, "Cyclic codes for error detection," Proc. IRE, vol. 49, pp. 228–235; January, 1961.
[8] ——, "Encoding and error-correction procedures for the Bose-Chaudhuri codes," IRE Trans. on Information Theory, vol. IT-6, pp. 459–470; September, 1960.

On the Optimum Range Resolution of Radar Signals in Noise*

N. J. NILSSON†, MEMBER, IRE

Summary—**Optimum radar resolution is recognized to be a problem in distinguishing between different possible target configurations. Radar reception systems which perform optimum range resolution are then designed using the principles of statistical decision theory. In particular, the design of the optimum resolution system is carried out for a squared-error loss function, modified to provide extra penalties for wrong guesses about the number of targets present. Such a system is capable of simultaneously deciding the number of targets present, their spatial positions (ranges) and their relative amplitudes. The analysis also includes a discussion of an optimum device for the resolution of distributed (clutter-like) targets.**

Introduction

THE ability to resolve multiple-echo signals in time determines the range resolution of radar systems. The resolution rule-of-thumb for pulse radars is that echo pulses separated by a pulse length can be resolved, but echo pulses which overlap to any significant extent appear as only one target. Woodward[1] has developed a generalization of this rule which is applicable even for radar (sounding) signals whose time-bandwidth products are larger than unity. After defining the so-called Radar Ambiguity Function, Woodward infers that the time resolution cell for *any* signal is equal to the reciprocal of the (sounding) signal bandwidth.

However, this classical definition of resolution takes only qualitative account of the fact that the signals are embedded in noise. Even very narrow-band signals should be resolvable for arbitrarily small time separations in the complete absence of noise. We intuitively expect, then, that our ability to resolve two or more known signals in noise should depend, not only on the signal bandwidth, but on the echo-to-noise power ratio. In this paper, we shall treat the resolution problem as a problem of combined signal detection and signal estimation. Systems which achieve optimum range resolution will be derived, and their characteristics compared with those systems which are optimum only in the single-target detection sense.

Various authors have recognized the need for treating the general signal resolution problem in a more precise fashion than can be done by appeals to Woodward's ambiguity function alone. Swerling[2] has analyzed the problem of resolving two radar targets at the same range, but at (slightly) different angles within the antenna beamwidth. Helstrom[3,4] discusses the problem of distinguishing between two noise-corrupted signals whose form and location are known exactly. In this paper, we shall attempt first to formulate a suitable definition of radar resolution and then to apply this definition to the design of systems which perform optimum radar range resolution.

What is meant by "radar resolution?" A radar system achieving "good" resolution should be able to provide continuous and reliable answers to the following four questions which specify the target configuration:

a) How many (point) targets are there?
b) What are their relative (spatial) positions?
c) What are their relative velocities?
d) What are their relative amplitudes (cross sections)?

We define resolution in the following way. *An ensemble of target configurations (specified by the set of all situations which are allowable answers to the above four questions) is radar resolvable with average loss £, relative to a certain level of additive noise, a certain sounding signal, and a certain loss function L, if their respective composite echo returns can be distinguished by the Bayes decision device with minimum average loss £.*

Distinguishing among the many different possible target configurations is a problem in statistical decision theory. After defining the loss function L incurred for wrong guesses about the target configuration, we may calculate the average loss for any decision system. Decision systems with the lowest average loss are called Bayes decision systems and, therefore, according to the above definition, achieve the best resolution.

This definition of resolution prompts us to ask three more questions:

1) What *is* the optimum or Bayes radar resolving system relative to the noise, the sounding signal, and the loss function?

* Received by the PGIT, December 7, 1960. The research reported in this paper was conducted during the author's recently-completed term of active duty at Rome Air Dev. Ctr., Griffiss AFB, Rome, N. Y.

† Stanford Res. Inst., Menlo Park, Calif.

[1] P. M. Woodward, "Probability and Information Theory with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953.

[2] P. Swerling, "The resolvability of point sources," in "Proceedings of Symposium on Decision Theory and Applications to Electronic Equipment Development, Vol. I," Rome Air Dev. Ctr. Griffiss AFB, Rome, N. Y., RADC-TR-60-70A; April, 1960.

[3] C. W. Helstrom, "The resolution of signals in white Gaussian noise," Proc. IRE, vol. 43, pp. 1111–1118; September, 1955.

[4] C. W. Helstrom, "Statistical Theory of Signal Detection," Pergamon Press, New York, N. Y., ch. X; 1960.

2) How well does it resolve (what is the minimum average loss) relative to the noise and the sounding signal for the given loss function?
3) For what (which) particular sounding signal(s) is the resolution best?

The present paper is devoted to finding the optimum system asked for in question 1). The optimum system should provide answers to questions a)–d) about the target configuration and distinguish optimally between different target configurations. We shall simplify the analysis, however, by assuming that all targets have the same angle and have zero velocity; that is, we inquire only about the number of targets, their relative ranges and their relative amplitudes. Questions 2) and 3), which have to do with quality of resolution and optimum sounding signal selection, are beyond the scope of this paper. Future treatment of these interesting questions should provide answers which will replace the rule-of-thumb about the reciprocal of the signal bandwidth with much more precise statements about resolution quality.[5]

Bayes Decision Procedures

The Target Density Function

When a radar (sounding) signal of form $s(t)$ is transmitted into a target environment, the received echo will be a linear superposition of like signals. In this paper, it will be assumed that the targets are stationary so that the individual echo returns comprising the total received echo will differ only in time of arrival and amplitude, corresponding to different target ranges and cross sections. The total received echo signal $S(t)$ can then be written as

$$S(t) = \int_0^T A(\tau)s(t - \tau)\, d\tau \qquad 0 \leq t \leq T \qquad (1)$$

where

$A(\tau)$ = target density function $0 \leq \tau \leq T$
$s(t)$ = transmitted (sounding) signal
T = maximum possible target range (in seconds), assumed to be much greater than the reciprocal of the sounding signal bandwidth.

$s(t)$ is normalized such that

$$\int_0^T s^2(t)\, dt = 1. \qquad (2)$$

$A(\tau)$ describes everything that is relevant about a stationary target configuration. Over that range of τ where a distributed target may exist, $A(\tau)$ is a continuous function of τ. On the other hand, point targets are represented by Dirac delta functions. For example, if there is a point target of amplitude A_1 at range τ_1, and another of amplitude A_2 at range τ_2, then

$$A(\tau) = A_1\, \delta(\tau - \tau_1) + A_2\, \delta(\tau - \tau_2)$$

and from (1)

$$S(t) = A_1 s(t - \tau_1) + A_2 s(t - \tau_2).$$

In what follows, we may at times restrict the possible target configurations to some given set. Let us call the set of allowed target configurations $\mathfrak{A}$ and let A stand for any member $A(\tau)$ of the set $\mathfrak{A}$. Suppose further that we know (or may ascribe) some *a priori* probability measure $p[A]$ to each member A of $\mathfrak{A}$. Such a probability measure is necessary in the Bayes decision procedure, and it is well to state at the outset our assumptions about $p[A]$, even though they be of questionable merit. Sometimes we shall also define a probability measure $p[S]$ on the members S in the set $\mathfrak{S}$ of all possible received echos. S, of course, stands for $S(t)$. When there is a one-to-one correspondence between the members of sets $\mathfrak{A}$ and $\mathfrak{S}$, then $p[A]$ will be identical with $p[S]$.

The Received Datum

We shall assume that the composite echo $S(t)$ is accompanied by additive, stationary, Gaussian noise denoted by $N(t)$. For simplicity, let $N(t)$ have zero mean value. The total received waveform is then

$$X(t) = S(t) + N(t) \qquad 0 \leq t \leq T. \qquad (3)$$

We shall use the notation X as representing an arbitrary received waveform $X(t)$ belonging to some set of waveforms x.

Loss Functions and Bayes Decisions

The problem posed in this paper is: after reception of X, we must decide in an "optimum" manner which A in $\mathfrak{A}$ represents the actual target configuration. We shall denote the result of this decision as $\hat{A}$, our estimate of A. In this paper, we shall equate the set $\hat{\mathfrak{A}}$ of all possible estimates with the set $\mathfrak{A}$ of all possible target density functions. That is, we shall never make an estimate $\hat{A}$ corresponding to an impossible target configuration.

We have defined an "optimum" decision as a Bayes decision. Let us denote the Bayes decision for A as $\hat{A}_b$. To make a Bayes decision, we must define a loss function L which fixes the loss incurred for erroneous decisions. That is, if A represents the actual target situation, but we decide $\hat{A}$, then we lose an amount $L[\hat{A}, A]$. The Bayes estimate $\hat{A}_b$ is that $\hat{A}$ which minimizes the average value of L.[6] Since choice of any $\hat{A}$ also implies by (1) a composite echo $S(\hat{A})$, we may choose to define our loss function in terms of $S(\hat{A})$ and S, that is, $L = L[S(\hat{A}), S]$. [When it

[5] For an interesting study of pairwise resolution and a technique for providing some answers to the problem of pairwise resolution quality, see G. W. Preston, "The Advanced Theory of Radar Measurements," Final Rept. to the Rome Air Dev. Ctr. on Contract AF 30(602)-2120, General Atronics Corp. Rept. No. 799-207-12; August 20, 1960.

[6] For a general discussion of Decision Theory, see: D. Blackwell and M. A. Girshick, "Theory of Games and Statistical Decisions," John Wiley and Sons, Inc., New York, N. Y.; 1954. For applications of Decision Theory to signal detection, see: D. Middleton, "Random processes, signals, and noise—an introduction to statistical communication theory," in "Pure and Applied Physics, Introductory Series," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 21; 1960.

is convenient to write $S(\hat{A})$ explicitly as a function of time, we shall denote it by $\hat{S}(t)$.]

It is obvious that one way to minimize the average value of L is to choose $\hat{A}$ as some function of X in such a way that for *every* X, L averaged over the *a posteriori* probability measure for A, denoted by $p[A \mid X]$, is minimized.[7] That is, minimize the conditional expectation of L given X, denoted by $E_X[L]$. If L is defined in terms of $\hat{A}$ and A:

$$E_X[L(\hat{A}, A)] = \sum_{\mathfrak{A}} p[A \mid X]L[\hat{A}, A] \tag{4}$$

where $p[A \mid X]$, an *a posteriori* probability measure, is called the *a posteriori* likelihood of A being the target density function given X. The sum over the set $\mathfrak{A}$ represents an average over this set. $p[A \mid X]$ is obtained from the defined *a priori* probability measure $p[A]$ using Bayes' rule and the noise statistics. In case L is defined in terms of $S(\hat{A})$ and S, and, if there is a one-to-one correspondence between members of the sets $\mathfrak{A}$ and $\mathfrak{S}$, then we may write

$$E_X[L\{S(\hat{A}), S\}] = \sum_{\mathfrak{S}} p[S \mid X]L[S(\hat{A}), S] \tag{5}$$

where $p[S \mid X]$, an *a posteriori* probability measure, is called the *a posteriori* likelihood of the composite echo S given X. $p[S \mid X]$ can be calculated from the probability measure $p[S]$ using Bayes' rule and the noise statistics. In both (4) and (5), $E_X[L]$ is a function of $\hat{A}$. The Bayes estimate A_b minimizes $E_X[L]$ over all other estimates $\hat{A}$ in the set of possible target configurations $\mathfrak{A}$. In order to proceed further to see what sort of decision procedures arise, we must assume some particular loss functions.

Estimation of the Parameters of n Point Targets

The Loss Function

Before trying to design a system to guess how many targets exist, let us assume that a known number n exists and that we must estimate their parameters (ranges and amplitudes). Let us define the loss function L as being the integrated squared error in terms of the echo signal.

$$L = \int_0^T \{\hat{S}(t) - S(t)\}^2 \, dt \tag{6}$$

where $S(t)$ is the actual echo signal (a random process), and $\hat{S}(t)$ is the composite echo signal implied in (1) by the choice of the estimate $\hat{A}(\tau)$. $\hat{S}(t)$ is therefore a function of the estimate $\hat{A}$. Let us also restrict the set $\mathfrak{A}$ to those target density functions which represent a collection of n point targets, that is

$$A(\tau) = \sum_{i=1}^{n} A_i \, \delta(\tau - \tau_i) \tag{7}$$

where

A_i = amplitude of ith target.
τ_i = range (in seconds) of ith target, $\tau_i \leq T$ for all i.

[7] Middleton, *op. cit.*, p. 1028.

Estimation of a particular target density function $\hat{A}$ is now achieved by estimating the components of the n-vectors

$$\hat{\boldsymbol{\tau}} = [\hat{\tau}_1, \hat{\tau}_2, \cdots \hat{\tau}_n] \quad \text{and} \quad \hat{\mathbf{A}} = [\hat{A}_1, \hat{A}_2, \cdots \hat{A}_n].$$

Making use of (6) and (7), and assuming a one-to-one correspondence between the members of sets $\mathfrak{A}$ and $\mathfrak{S}$, the conditional expectation of the loss is

$$E_X\left[\int_0^T \left\{\sum_{i=1}^{n} \hat{A}_i s(t - \hat{\tau}_i) - S(t)\right\}^2 dt\right]$$

where the $\hat{A}_i$ and $\hat{\tau}_i$ are estimates of A_i and τ_i, respectively, and the expectation is taken over the *a posteriori* likelihood for the waveform S. Expansion of this expression yields

$$\begin{aligned} E_X[L] = {} & \int_0^T \left[\sum_{i=1}^{n} \hat{A}_i s(t - \hat{\tau}_i)\right]^2 dt \\ & - 2\int_0^T \left[\sum_{i=1}^{n} \hat{A}_i s(t - \hat{\tau}_i)\right] E_X[S(t)] \, dt \\ & + \int_0^T E_X[S^2(t)] \, dt. \end{aligned} \tag{8}$$

Since $E_X[L]$ is an average taken over the *a posteriori* likelihood for $S(t)$, the term $E_X[S(t)]$ in the above equation will be a waveform or function of time.

Minimization of the Average Loss

Since we are trying to minimize $E_X[L]$ by choice of $\hat{A}_i$ and $\hat{\tau}_i (i = 1, 2, \cdots n)$, we need only maximize the expression

$$\begin{aligned} J_n = {} & 2\sum_{i=1}^{n} \hat{A}_i \int_0^T E_X[S(t)]s(t - \hat{\tau}_i) \, dt \\ & - \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{A}_i\hat{A}_j \int_0^T s(t - \hat{\tau}_i)s(t - \hat{\tau}_j) \, dt. \end{aligned} \tag{9}$$

The above equation can be written as

$$J_n = 2\sum_{i=1}^{n} \hat{A}_i\hat{\phi}_i - \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{A}_i\hat{A}_j\hat{\lambda}_{ij} \tag{10}$$

where

$$\hat{\phi}_i = \phi(\hat{\tau}_i) = \int_0^T E_X[S(t)]s(t - \hat{\tau}_i) \, dt$$

and

$$\hat{\lambda}_{ij} = \lambda(\hat{\tau}_i - \hat{\tau}_j) = \int_0^T s(t - \hat{\tau}_i)s(t - \hat{\tau}_j) \, dt$$

$$= \text{the "autocorrelation function" of } s(t).$$

$\hat{\phi}_i$ is obtained as the output of a filter, at time $\hat{\tau}_i$, which is matched to the transmitted signal $s(t)$ when $E_X[S(t)]$ is the filter input waveform.

We shall first maximize J_n by proper choice of the $\hat{A}_i$ and then obtain final maximization by choice of the $\hat{\tau}_i$. We can differentiate (10) with respect to $\hat{A}_k$ and set the

result equal to zero to derive the following relationship that must hold for the maximizing amplitude estimates $\hat{A}_{io}$:

$$\sum_{i=1}^{n} \hat{A}_{io}\hat{\lambda}_{ki} = \hat{\phi}_k. \tag{11}$$

If both sides of the above equation are multiplied by $\hat{A}_{ko}$ and summed over k, we obtain

$$\sum_{i,k=1}^{n} \hat{A}_{io}\hat{A}_{ko}\hat{\lambda}_{ki} = \sum_{k=1}^{n} \hat{A}_{ko}\hat{\phi}_k. \tag{12}$$

Since (12) must also be satisfied by the maximizing estimates $\hat{A}_{io}$, then J_n, maximized over the $\hat{A}_i$, can be written from (10) as

$$J_{no} = \sum_{i=1}^{n} \hat{A}_{io}\hat{\phi}_i \tag{13}$$

where the $\hat{A}_{io}$ must satisfy (11).

It is perhaps more convenient to express the above relationships in matrix notation. Let us represent all the $\hat{A}_{io}$ by the the n-dimensional column vector

$$\hat{\mathbf{A}}_0 = \begin{bmatrix} \hat{A}_{1o} \\ \hat{A}_{2o} \\ \vdots \\ \hat{A}_{io} \\ \vdots \\ \hat{A}_{no} \end{bmatrix} \tag{14}$$

and group all the $\hat{\lambda}_{ij}$ into an $n \times n$ matrix

$$\hat{\lambda} = \begin{bmatrix} \hat{\lambda}_{11} & \hat{\lambda}_{12} & \cdots & \hat{\lambda}_{1j} & \cdots & \hat{\lambda}_{1n} \\ \hat{\lambda}_{21} & & & & & \\ \vdots & & & & & \\ \hat{\lambda}_{i1} & \cdots & & \hat{\lambda}_{ij} & \cdots & \hat{\lambda}_{in} \\ \vdots & & & & & \vdots \\ \hat{\lambda}_{n1} & & \cdots & & & \hat{\lambda}_{nn} \end{bmatrix}. \tag{15}$$

Then the set of (11) may be written as the matrix equation

$$\hat{\boldsymbol{\phi}} = \hat{\lambda}\hat{\mathbf{A}}_o \tag{16}$$

where $\hat{\boldsymbol{\phi}}$ is the n-dimensional column vector

$$\hat{\boldsymbol{\phi}} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_i \\ \vdots \\ \hat{\phi}_n \end{bmatrix}. \tag{17}$$

Both $\hat{\boldsymbol{\phi}}$ and $\hat{\lambda}$ are functions of the vector $\hat{\boldsymbol{\tau}} = [\hat{\tau}_1, \hat{\tau}_2, \cdots \hat{\tau}_n]$. The maximizing estimate $\hat{\mathbf{A}}_o$ must then satisfy the matrix equation

$$\hat{\mathbf{A}}_o = \hat{\lambda}^{-1}\hat{\boldsymbol{\phi}} \tag{18}$$

where $\hat{\lambda}^{-1}$ is the inverse of $\hat{\lambda}$.

J_{no} can also be written from (13) in vector notation as the dot product

$$J_{no} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{A}}_o \tag{19}$$

or, using the relationship given in (18),

$$J_{no} = \hat{\boldsymbol{\phi}} \cdot (\hat{\lambda}^{-1}\hat{\boldsymbol{\phi}}). \tag{20}$$

Eq. (20) is an expression for J_n maximized over the vector $\hat{\mathbf{A}}$. J_{no} is still a function of the vector $\hat{\boldsymbol{\tau}}$. J_{no} must then be further maximized by choice of a $\hat{\boldsymbol{\tau}}_b$ which fixes $\hat{\boldsymbol{\phi}}_b$ and $\hat{\lambda}_b{}^{-1}$.

The final maximum for J_n can then be written as

$$J_{nb} = \hat{\boldsymbol{\phi}}_b \cdot (\hat{\lambda}_b{}^{-1}\hat{\boldsymbol{\phi}}_b). \tag{21}$$

Upon discovering the maximizing $\hat{\boldsymbol{\tau}}_b$, we may write an expression for the Bayes estimate $\hat{\mathbf{A}}_b$ from (18) in the following way

$$\hat{\mathbf{A}}_b = \hat{\lambda}_b^{-1}\hat{\boldsymbol{\phi}}_b. \tag{22}$$

Let us summarize the above expressions by stating the rule for finding the Bayes estimates $\hat{A}_{1b} \cdots \hat{A}_{nb}$, $\hat{\tau}_{1b}, \cdots \hat{\tau}_{nb}$ for the positions and amplitudes of a known number n of targets:

1) Form the quantity $\phi(\tau)$ by passing $E_X[S(t)]$ through a filter matched to $s(t)$.

2) Form an n-vector $\hat{\boldsymbol{\phi}} = [\phi(\hat{\tau}_1), \cdots \phi(\hat{\tau}_n)]$ and an $n \times n$ matrix $\hat{\lambda}$ by selecting estimates $\hat{\tau}_1, \cdots \hat{\tau}_n$ in such a way that the quadratic form $J_{no} = \hat{\boldsymbol{\phi}} \cdot (\hat{\lambda}^{-1}\hat{\boldsymbol{\phi}})$ is maximized. Let us say that maximization occurs for the vector $\hat{\boldsymbol{\tau}}_b = [\hat{\tau}_{1b}, \cdots \hat{\tau}_{nb}]$ which determines a $\hat{\boldsymbol{\phi}}_b$ and a $\hat{\lambda}_b$.

3) Using the maximizing $\hat{\boldsymbol{\tau}}_b$, calculate the Bayes amplitude estimates $\hat{A}_{1b}, \cdots \hat{A}_{nb}$ by

$$\hat{\mathbf{A}}_b = \hat{\lambda}_b^{-1}\hat{\boldsymbol{\phi}}_b.$$

In short, the receiver has only to calculate the conditional mean $E_X[S(t)]$ and pass this waveform through a matched filter to obtain $\phi(\tau)$. Then, using its knowledge of $\lambda(\tau)$, it performs certain maximizing operations to obtain $\hat{\mathbf{A}}_b$ and $\hat{\boldsymbol{\tau}}_b$. If $S(t)$ is a T-second sample from a stationary Gaussian process,[8] $E_X[S(t)]$ is equal to that $S_o(t)$ which maximizes the *a posteriori* likelihood $p[S \mid X]$. (In a Gaussian distribution, the mode equals the mean.)

[8] The requirement that $S(t)$ must be a sample from a stationary process involves ignoring the radar "range-to-the-fourth-power" law for reasonable ensembles of target configurations. The assumption that $S(t)$ is Gaussian seems reasonable if the proper rationalizations about $s(t)$ and $p[A]$ are made. In particular, for any $s(t)$, S will be Gaussian if the A_i are Gaussian. However, the relationship between the statistics of $S(t)$ and $A(t)$ for various sounding signals should be investigated.

Youla[9] has shown that $S_o(t) = E_X[S(t)]$ is given by the following set of integral equations:

$$S_o(t) = \int_0^T X(\tau)h(t-\tau)\,d\tau, \qquad 0 \le t \le T$$

$$\int_0^T [R_S(\xi-\tau) + R_N(\xi-\tau)]h(t-\tau)\,d\tau = R_S(\xi - t) \quad (23)$$

where

$R_S(\tau)$ = autocovariance function of the random process of which $S(t)$ is a T-second sample

and

$R_N(\tau) = \langle N(t)N(t+\tau)\rangle$, the autocovariance function of the noise.

These two equations state that $S_o(t)$ can be obtained from a linear filter with input $X(t)$ and impulse response $h(\tau)$ where $h(\tau)$ is the solution to a modified Wiener-Hopf integral equation. As the equations stand, $h(\tau)$ is not physically realizable, but this situation can be corrected if we are willing to tolerate, at most, a delay of T seconds in obtaining $S_o(t)$. In the limit as T becomes very large, the filter specified in (23) approaches the unrealizable case of Wiener's least-mean-square-error filter. Finally, to obtain $\phi(\tau)$, $E_X[S(t)]$ is passed through a filter matched to $s(t)$ as shown in Fig. 1.

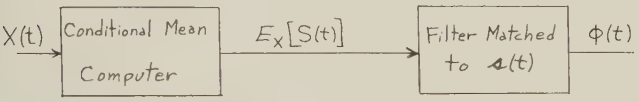

Fig. 1—Processing for $\phi(\tau)$.

Special Case: $n = 1$

When it is known that there is only one target present, the Bayes estimates for A_1 and τ_1 are obtained in a straightforward manner. $\hat{\lambda} = \hat{\lambda}^{-1} = (1)$ in the one-dimensional case so that, from (20),

$$J_{1,o} = \hat{\phi}_1^2. \quad (24)$$

That is, we must choose an estimate $\hat{\tau}_1$, which maximizes $\phi(\tau)$, and thus $\phi^2(\tau)$. Then, from (22)

$$\hat{A}_{1b} = \hat{\phi}_{1b}. \quad (25)$$

Referring to Fig. 2, we locate the maximum of $\phi(\tau)$ and equate the target strength with this maximum. Such a procedure using a matched filter operating on $X(t)$, instead of on $E_X[S(t)]$, has long been the accepted procedure.

[9] D. C. Youla, "The use of the method of maximum likelihood in estimating continuous-modulated intelligence which has been corrupted by noise," IRE Trans. on Information Theory, no. IT-3, pp. 90–105; March 1954.

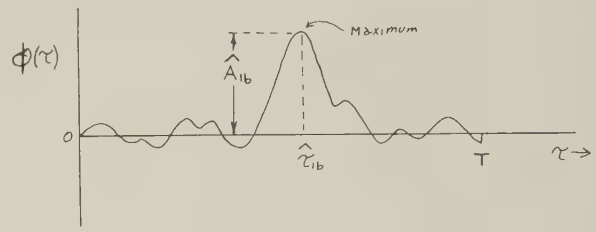

Fig. 2—Bayes estimate for a single target.

Special Case: $n = 2$

For two targets, we must make estimates $\hat{A}_{1b}$, $\hat{A}_{2b}$, $\hat{\tau}_{1b}$, and $\hat{\tau}_{2b}$. We first calculate $\hat{\lambda}^{-1}$ from $\hat{\lambda}$.

$$\hat{\lambda} = \begin{bmatrix} 1 & \hat{\lambda}_{12} \\ \hat{\lambda}_{12} & 1 \end{bmatrix} \quad (26)$$

where $\hat{\lambda}_{12} = \hat{\lambda}_{21}$ because $\lambda(\tau)$ is an even function. Inverting the above matrix yields

$$\hat{\lambda}^{-1} = \begin{bmatrix} \dfrac{1}{1-\hat{\lambda}_{12}^2} & \dfrac{-\hat{\lambda}_{12}}{1-\hat{\lambda}_{12}^2} \\ \dfrac{-\hat{\lambda}_{12}}{1-\hat{\lambda}_{12}^2} & \dfrac{1}{1-\hat{\lambda}_{12}^2} \end{bmatrix} \quad (27)$$

so that from (20)

$$J_{2,o} = \frac{\hat{\phi}_1^2 - 2\hat{\lambda}_{12}\hat{\phi}_1\hat{\phi}_2 + \hat{\phi}_2^2}{1-\hat{\lambda}_{12}^2}. \quad (28)$$

We must choose a $\hat{\tau}_{1b}$ and a $\hat{\tau}_{2b}$ such that the resulting $\hat{\phi}_{1b}$, $\hat{\phi}_{2b}$, and $\hat{\lambda}_{12b}$ maximize $J_{2,o}$. $\hat{\tau}_{1b}$ and $\hat{\tau}_{2b}$ are the Bayes estimates of the target positions. Then, using $\hat{\phi}_b$ and $\hat{\lambda}_b^{-1}$ in (22), we calculate

$$\hat{A}_{1b} = \frac{\hat{\phi}_{1b} - \hat{\lambda}_{12b}\hat{\phi}_{2b}}{1-\hat{\lambda}_{12b}^2} \quad (29)$$

and

$$\hat{A}_{2b} = \frac{\hat{\phi}_{2b} - \hat{\lambda}_{12b}\hat{\phi}_{1b}}{1-\hat{\lambda}_{12b}^2}.$$

Eq. (29) is identical with the result of Helstrom[4] who discusses a similar resolution problem, except that Helstrom uses $X(t)$ as the input to the matched filter instead of $E_X[S(t)]$.

Specialized Two-Target Configurations

It is seen that a generalization of the simple matched-filter is optimum when two targets of unknown positions and amplitudes are present. Maximization of (28) in two dimensions is somewhat more complex, though, than looking for the maximum output of a simple filter. $\hat{\phi}_{1b}$ and $\hat{\phi}_{2b}$ will not in general occur at relative maxima of $\phi(\tau)$ unless these relative maxima are very high indeed and are also separated by more than a correlation interval. (It is beyond the scope of this paper to discuss possible means of implementing a device for automatically computing $\hat{\tau}_{1b}$ and $\hat{\tau}_{2b}$; we shall limit ourselves to the task

of specifying mathematically the optimum operations.) Let us study some limiting-case two-target configurations to gain some insight into properties of the optimum procedure.

1) *Two Targets Known to be Separated by More Than the Correlation Interval*: If $|\tau_2 - \tau_1|$ is always greater than the reciprocal of the sounding signal bandwidth, $\hat{\lambda}_{12}$ will approach 0, and, from (28)

$$J_{2,o} = \hat{\phi}_1^2 + \hat{\phi}_2^2. \tag{30}$$

It is obvious that (30) can be maximized by choosing for $\hat{\phi}_{1b}$ and $\hat{\phi}_{2b}$ the two highest peaks in $\phi(\tau)$ which are separated by more than the correlation interval. Eq. (29), with $\hat{\lambda}_{12b} = 0$, is then used to compute $\hat{A}_{1b}$ and $\hat{A}_{2b}$. This technique is close to the way in which multitarget situations are handled by present-day radars.

2) *Two Targets Indistinguishably Close*: If it is known that $\tau_1 \cong \tau_2$, and it is only needed to find the mean range, the two-target estimation problem reduces to the one-target problem. As $\hat{\tau}_1 \to \hat{\tau}_2$, and thus $\hat{\lambda}_{12} \to 1$, and $\hat{\phi}_2 \to \hat{\phi}_1$, (28) becomes

$$J_{2,o} = \hat{\phi}_1^2 = \hat{\phi}_2^2 \tag{31}$$

which can be maximized by finding the maximum of $\phi(\tau)$.

3) *A Small Target Near a Large One*: If $A_1 \gg A_2$, we may be justified in setting $\hat{\phi}_{1b} = \max \phi(\tau)$, ignoring what we may later decide as estimates of τ_2 and the consequent $\hat{\lambda}_{12}$. Thus, $\hat{\tau}_{1b}$ is approximately the τ which maximizes $\phi(\tau)$. For any received waveform, this maximum will be some constant $\hat{\phi}_{1b}$.

Regarding $\hat{\phi}_1$ as a constant in (28), we may write

$$J_{2,o} = \hat{\phi}_{1b}^2 + \frac{[\hat{\phi}_2 - \lambda(\hat{\tau}_2 - \hat{\tau}_{1b})\hat{\phi}_{1b}]^2}{1 - \lambda^2(\hat{\tau}_2 - \hat{\tau}_{1b})}. \tag{32}$$

$J_{2,o}$ can be maximized by choice of $\hat{\tau}_2$ by selecting the τ which maximizes the quantity

$$\frac{[\phi(\tau) - \hat{\phi}_{1b}\lambda(\tau - \hat{\tau}_{1b})]^2}{1 - \lambda^2(\tau - \hat{\tau}_{1b})}.$$

The above expression describes approximately what the receiver must do to locate a small target in the presence of a large one. First, the maximum of $\phi(\tau)$ is found and its time of occurrence noted. The large target is guessed to be located at this point. Its effect is *subtracted* from $\phi(\tau)$, and the result is squared and divided by $[1 - \lambda^2(\tau - \hat{\tau}_{1b})]$. This new waveform is then scanned in τ for a maximum which occurs, say, at $\hat{\tau}_{2b}$. Now that $\hat{\tau}_{1b}$ and $\hat{\tau}_{2b}$ are known, (29) allows us to calculate estimates for the target amplitudes.

A Multitarget Detection and Estimation Problem

The Hybrid Loss Function and Its Minimization

One important assumption that we have made in our development must now be extended. So far, we have assumed a known number n of targets, making the problem one of estimation rather than one of detection. If n is random, we are faced with a combined detection and estimation problem, the solution of which will lead to optimum resolution systems as they were defined in the Introduction. We will also have to modify somewhat the loss function given by (6) so that extra penalties can result when the wrong *number* of targets is guessed.

Let us consider the following loss function, written with explicit reference to the number of targets guessed to be present and actually present:

$$L[S(\hat{A}), S; i, j] = \int_0^T [S(\hat{A}) - S(t)]^2\, dt + \alpha_{ij}. \tag{33}$$

In the above equation, the notation is the same as that in (6). The added term α_{ij} is the *extra* loss incurred for guessing i targets present when really there are j. Let us compose a matrix from the components α_{ij}:

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots & \\ \alpha_{10} & & & & \\ \vdots & & & & \\ \vdots & & & \alpha_{ij} & \cdots \\ \vdots & & & \vdots & \\ \vdots & & & \vdots & \end{pmatrix}. \tag{34}$$

$\boldsymbol{\alpha}$ is an infinite matrix with elements α_{ij} defined for all $i = 0, 1, 2, \cdots$, and $j = 0, 1, 2, \cdots$.

Proceeding exactly as in the previous section, we want to minimize the conditional expectation of the loss, given $X(t)$. It is easily shown that this minimization is equivalent to maximizing the expression

$$K_n = J_n - E_X[\alpha_{nj}] \tag{35}$$

by simultaneous choice of n, $\hat{\boldsymbol{\tau}}$ and $\hat{\mathbf{A}}$; where J_n is given, for each n, by (10), and $E_X[\alpha_{nj}]$ = the expectation given $X(t)$ of α_{nj} over all possible numbers of targets j.

The following procedure is used to maximize K_n. First, for each n, $J_{nb} = \hat{\boldsymbol{\phi}}_b \cdot (\hat{\boldsymbol{\lambda}}_b^{-1}\hat{\boldsymbol{\phi}}_b)$ is found by selecting an n-dimensional vector $\hat{\boldsymbol{\tau}}_b$ such that the resulting $\hat{\boldsymbol{\phi}}_b$ and $\hat{\boldsymbol{\lambda}}_b^{-1}$ maximize the n-quadratic form $\hat{\boldsymbol{\phi}} \cdot (\hat{\boldsymbol{\lambda}}^{-1}\hat{\boldsymbol{\phi}})$. Then we calculate

$$\begin{aligned} E_X[\alpha_{nj}] &= \alpha_{n0}p_X(0) + \alpha_{n1}p_X(1) \\ &\quad + \cdots \alpha_{nj}p_X(j) + \cdots \\ &= \boldsymbol{\alpha}_n \cdot \mathbf{p}_X \end{aligned} \tag{36}$$

where

$\boldsymbol{\alpha}_n$ = a Hilbert space vector with components α_{n0}, α_{n1}, α_{n2}, $\cdots$

$\mathbf{p}_X$ = a Hilbert space vector with components $p_X(0)$, $p_X(1)$, $p_X(2)$, $\cdots$

and

$p_X(i)$ = the *a posteriori* probability that i targets are present given the received waveform $X(t)$.

By using J_{nb} instead of J_n in (35), we have the maximized expression

$$K_{no} = J_{nb} - \boldsymbol{\alpha}_n \cdot \mathbf{p}_X \tag{37}$$

which must be further maximized by choice of $\hat{n}$. If it were not for the term $\boldsymbol{\alpha}_n \cdot \mathbf{p}_X$, K_{no} would have no maximum over n because it can be shown that $J_{n+1,\,b} \geq J_{nb}$ for all b.[10] In practice, the highest value of n to be tested will be some finite number, probably two or three, so that all the K_{no} could, in principle, be calculated and the largest found. Once $\hat{n}$ is determined, the $\hat{n}$-vector $\hat{\tau}_b$ gives the Bayes range estimates, and the corresponding $\hat{\boldsymbol{\phi}}_b$ and $\hat{\boldsymbol{\lambda}}_b^{-1}$ are used to calculate $\hat{\mathbf{A}}_b$.

Let us now review the procedure for optimum estimation and detection in a multitarget environment:

1) The receiver calculates $\phi(\tau)$ and $\mathbf{p}_X$ from the received waveform $X(t)$. $\phi(\tau)$ is again the matched-filtered version of $E_X[S(t)]$. The calculation of $E_X[S(t)]$ for the case of random n will be discussed later in greater detail. $\mathbf{p}_X$ is a vector whose ith component is just the *a posteriori* probability that i targets are present.

2) From $\phi(\tau)$ and $\lambda(\tau)$ the receiver finds, for each n, that $\hat{\boldsymbol{\phi}}_b$ and $\hat{\boldsymbol{\lambda}}_b^{-1}$ which maximize the n-quadratic form $\hat{\boldsymbol{\phi}} \cdot (\hat{\boldsymbol{\lambda}}^{-1} \hat{\boldsymbol{\phi}})$. The maxima of the n-quadratic forms are called J_{nb}.

3) The quantity $K_{no} = J_{nb} - \boldsymbol{\alpha}_n \cdot \mathbf{p}_X$ is the then calculated and maximized over n. That is, if $K_{\hat{n}o} > K_{no}$ for all integers $n \neq \hat{n}$, then $\hat{n}$ is an optimum or Bayes estimate for the number of targets present.

4) The range estimates of the $\hat{n}$ targets are then the components of the $\hat{n}$-vector $\hat{\boldsymbol{\tau}}_b$ which determined the maximizing $\hat{\boldsymbol{\phi}}_b$ and $\hat{\boldsymbol{\lambda}}_b^{-1}$ for the $\hat{n}$th dimension.

5) The $\hat{n}$-dimensional vector $\hat{\boldsymbol{\phi}}_b$ and the $\hat{n} \times \hat{n}$ matrix $\hat{\boldsymbol{\lambda}}_b$ are then used to calculate the Bayes amplitude estimates $\hat{A}_{1b}, \cdots \hat{A}_{\hat{n}b}$ of the $\hat{n}$ targets by the expression $\hat{\mathbf{A}}_b = \hat{\boldsymbol{\lambda}}_b^{-1} \hat{\boldsymbol{\phi}}_b$. The above procedure completes the combined Bayes detection and estimation technique. For the loss function assumed, it provides optimum resolution. Special examples of this procedures will be considered below.

Calculation of $E_X[S(t)]$

When the number of targets n is a random variable, the method of Youla[9] cannot be directly applied to the calculation of $E_X[S(t)]$. When n is random, we may not assume that the *a priori* probability density function for the waveform $S(t)$ is a multidimensional Gaussian distribution if there is a finite possibility that $S(t) = 0$, *i.e.*, no targets. If the *a posteriori* probability for zero targets, given $X(t)$, is $p_X(0)$, and if *otherwise* $p[S]$ and $p[S \mid X]$ can be considered multidimensional Gaussian likelihood functions, then

$$E_X[S(t)] = [1 - p_X(0)] S_o(t) \tag{38}$$

where $S_o(t)$ is that S which maximizes the "continuous" or Gaussian portion of $p[S \mid X]$. $S_o(t)$ can be obtained by the same linear filter described in (23).

[10] For any n, a J_{n+1} which equals J_{nb} can always be obtained by choosing two of the $\hat{\tau}_i$ to be equal in the $(n+1)$-vector $\hat{\tau}$ to make it in reality an n-vector. Thus, $J_{n+1,\,b} \geq J_{nb}$.

Some Special Cases

Let us now consider some examples to illuminate the combined detection and estimation theory that has been developed. First, we must select an appropriate matrix $\boldsymbol{\alpha}$. Following somewhat the philosophy of Bennion,[11] who uses a loss function which penalizes a given amount for false alarms but penalizes only with the squared error for false rest, we write

$$\boldsymbol{\alpha} = \alpha \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 3 & 2 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}. \tag{39}$$

That is, we lose an extra amount 3α if we say 3 targets present when there are really none, etc.

1) *Detection of a Single Target*: Suppose we need only choose the larger of $K_{0,o}$ or $K_{1,o}$ and, if $K_{1,o}$ is our choice, make estimates of the single target's range and amplitude. Such a situation is a combined single-target detection and estimation problem. From (10), guessing no target present implies $J_{o,b} = 0$, so that

$$K_{0,o} = -\boldsymbol{\alpha}_o \cdot \mathbf{p}_X$$

and

$$K_{1,o} = \max_{\tau} \phi^2(\tau) - \boldsymbol{\alpha}_1 \cdot \mathbf{p}_X. \tag{40}$$

We must choose the maximum, which leads to the rule: say a target is present if

$$K_{1,o} - K_{0,o} = \{\max_{\tau} \phi^2(\tau) - [\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0] \cdot p_X\} > 0 \tag{41}$$

or if

$$\max_{\tau} \phi^2(\tau) > \alpha p_X(0)$$

otherwise say no target is present. If a target is declared present, $\hat{\tau}_{1b}$ and $\hat{A}_{1b}$ are calculated as before. Notice in this example that the combined detection and estimation problem involves essentially a threshold detection scheme even though the value of the threshold is a function of the received waveform [through $p_X(0)$].

2) *Pairwise Resolution*: To answer the question: "Are there two targets present or just one?" merely involves choosing between $K_{1,o}$ and $K_{2,o}$. Now the $J_{2,b}$ and $J_{1,b}$ are calculated using $\phi(\tau)$ and $\lambda(\tau)$. If

$$J_{2,b} - J_{1,b} > \alpha[p_X(1)] \tag{42}$$

then say two targets are present, but if not, then say only one target is present. Estimates for A_i and τ_i can then be made. If a small target is to be detected in the presence of a large one, the procedure under (32) may be

[11] D. Bennion, "Some Results in the Estimation of Signal Parameters," Stanford Electronics Lab., Stanford University, Stanford, Calif., Tech Rept. No. 10; September 10, 1956.

used to obtain the rule: say two targets are present if

$$\max_{\tau} \frac{[\phi(\tau) - \hat{\phi}_{1b}\lambda(\tau - \hat{\tau}_{1b})]^2}{1 - \lambda^2(\tau - \hat{\tau}_{1b})} > \alpha p_X(1) \tag{43}$$

otherwise say only one target present. The estimates for A_i and τ_i are obtained as before.

3) *Widely Separated Targets*: If it is known *a priori* that any targets will be widely separated, $\hat{\lambda}$ is an identity matrix as is $\hat{\lambda}^{-1}$. In this case, $J_{nb} = \sum_{i=1}^{n} \hat{\phi}_{ib}^2$ and

$$K_{no} = \sum_{i=1}^{n} \hat{\phi}_{ib}^2 - \boldsymbol{\alpha}_n \cdot \mathbf{p}_X. \tag{44}$$

The following procedure may be used to maximize $K_{n,o}$ by choice of n:

a) The receiver computes $\phi(\tau)$ and finds its maximum, ϕ_{1b}. If $\hat{\phi}_{1b}^2 > \alpha p_X(0)$, a target is announced and its parameters are computed.

b) The next highest peak in $\phi(\tau)$ which is more distant from τ_{1b} then a correlation interval is selected as $\hat{\phi}_{2b}$. If $\hat{\phi}_{2b}^2$ is greater than $\alpha[p_X(0) + p_X(1)]$, then a second target is announced and its parameters are computed.

c) This process is repeated until the threshold condition is not satisfied, giving a decision of, say, $\hat{n}$ targets and their parameters.

Discussion

We have proposed an optimum technique for simultaneously performing the generalized detection (how many targets) and the estimation (where are they and what are their amplitudes) problems. The detection problem involves the maximization of a set of quadratic forms ($n = 0, 1, 2, \cdots$). From each maximized quadratic form, a constant is subtracted which is a function of the *a posteriori* probability of a certain number of targets being present. Then, the resulting terms are compared to find the largest. We have assumed a modified squared-error loss function which inflicts extra penalties for false guesses about the number of targets present.

The receiver must compute the conditional expectation of the composite echo signal as well as the *a posteriori* probabilities for various numbers of targets being present. The conditional expectation of the echo signal can sometimes be obtained by a modified type of Wiener filtering operation on the received, noise-corrupted, waveform. If no assumptions about *a priori* echo statistics are made, the received datum (signal plus noise) is usually used in place of the conditional expectation of the composite echo signal. Such practice is a consequence of using a maximum likelihood procedure instead of a squared-error Bayes procedure.

So far, we have concerned ourselves only with point targets which cause $A(\tau)$ to take the form of delta functions. If optimum estimates of $A(\tau)$ for distributed targets, such as clutter, are to be found, a somewhat different tack must be taken. Now we shall discuss some results for this important problem.

Distributed Targets

The Turin Filter

Suppose $A(\tau)$ does not consist of point targets, but, instead, represents distributed targets such as clutter. Then a loss function involving the integrated squared error between $A(\tau)$ and the estimate $\hat{A}(\tau)$ might be appropriate. Such a problem was considered by Turin,[12] although he restricted his estimator to a linear filter.[13] Under the assumption that $A(\tau)$ is a T-second sample from a stationary Gaussian random process with autocovariance function $R_A(\tau)$, the transfer function of the optimum linear estimator, as derived by Turin is

$$T(\omega) = \frac{F^*(\omega)}{F(\omega)F^*(\omega) + \dfrac{G_N(\omega)}{G_A(\omega)}} \tag{45}$$

where

$F(\omega)$ = Fourier transform of $s(t)$
$F^*(\omega)$ = conjugate of $F(\omega)$
$G_N(\omega)$ = power spectral density of $N(t)$
= Fourier transform of $R_N(\tau)$
$G_A(\omega)$ = power spectral density function of the random process of which $A(t)$ is a T-second sample
= Fourier transform of $R_A(\tau)$.

Special Properties of the Turin Filter

The filter shown in Fig. 3 is a "crispening filter."

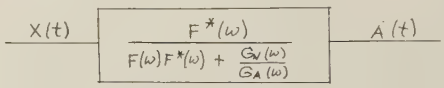

Fig. 3—Turin filter. The optimum linear estimator for the target density function.

(We disregard the extra time delay needed to insure realizability.) The $F^*(\omega)$ in the numerator "compresses" the received waveform by causing phase reinforcement, while the denominator accents the high-frequency components allowing for fast risetimes. For example, if the average noise power is much less than the average echo power, the Turin filter approaches an inverse filter. An inverse filter has a delta function output every time $s(t)$ occurs at the input. On the other hand, if the average noise power is much greater than the average echo power, then the Turin filter becomes a simple matched filter. No rise-time crispening can be done in the face of so much noise. In any case, if $|F(\omega)|$ is rectangular and if the noise and $A(\tau)$ are white, then the filter also becomes a matched filter.

[12] G. L. Turin, "On the estimation in the presence of noise of the impulse response of a random, linear filter," IRE Trans. on Information Theory, vol. IT-3, pp. 5–10; March, 1957.

[13] Which is no restriction under the assumption that $A(\tau)$ is Gaussian.

The resolution enhancing properties of this filter come about due to its crispening action. The lower the noise-to-echo power ratio, the greater is the crispening action, and hence, the finer is the potential range (time) resolution capability. It is assumed that $G_N(\omega)$ is known, and $G_A(\omega)$ can be calculated from the statistics of the ensemble of target density functions.

Conclusion

The problem considered in this paper, briefly stated, concerned optimum methods of signal detection. The signal form, however, was not known exactly except for the fact that is was a composite of known signal forms echoing from a random configuration of targets. We stated in the Introduction that the solution of such a detection problem also solves the optimum resolution problem if resolution is defined to mean distinguishing between the different possible composite echos. Various procedures were derived for making estimates of the target density function (which may either be discrete or continuous) under special assumptions regarding *a priori* probabilities and loss functions.

No attempt was made to evaluate different radar systems, including optimum ones, to determine just how well they are able to distinguish between different target configurations in the face of random noise. This problem was posed as question 2) in the Introduction and must be answered before quantitative statements can be made about resolution ability. Also unanswered is the problem of deciding what sounding signal to transmit so that resolution ability can be further enhanced. It is hoped that the present study might provide stimulation for a complete set of answers to these questions. In addition to the above two important problems, the following suggestions are offered as topics for further research:

1) Investigation of the *a priori* and *a posteriori* statistics of $S(t)$ for various assumptions about the statistics of the target configuration, the statistics of the noise, and the known sounding signal.
2) Generalization of the results of this paper to include the other spatial dimensions (angle) and velocity.
3) Investigation of other loss functions.
4) Instrumentation of a two-target resolver using (28) and measurement of its performance in noise.
5) Analysis of methods to calculate approximately the *a posteriori* probabilities for the number of targets present.

Appendix

List of Symbols

A, $A(\tau)$ = Target density function
A_i = Amplitude of ith target
$\hat{\mathbf{A}}$ = Target amplitude vector whose ith component is A_i
$\mathcal{A}$ = Set of all possible $A(\tau)$
$E_X[\ ]$ = Conditional expectation of $[\ ]$
$F(\omega)$ = Fourier transform of $s(t)$
$G_A(\omega)$, $G_N(\omega)$ = Power spectral density functions of $A(\tau)$ and $N(t)$, respectively
J_{no}, J_{nb} = The quadratic form $\hat{\boldsymbol{\phi}} \cdot (\hat{\boldsymbol{\lambda}}^{-1}\hat{\boldsymbol{\phi}})$ and its maximum, respectively
K_{no} = An expression derived from J_{nb} by subtraction of $\boldsymbol{\alpha}_n \cdot \mathbf{p}_X$
L = The loss function defined for each combination of $[\hat{A}(\tau), A(\tau)]$ or $[S(\hat{A}), S(t)]$
n = Number of point targets present
N, $N(t)$ = Zero-mean, stationary, Gaussian noise
$p[\ ]$, $p[\ |\ X]$ = Likelihood and conditional likelihood, respectively, of the waveform $[\ ]$
$p_X(i)$ = Conditional probability of i targets present given X
$\mathbf{p}_X$ = A Hilbert space vector whose ith component is $p_X(i)$
$R_A(\tau)$, $R_N(\tau)$, $R_S(\tau)$ = Autocovariance functions of the stationary random processes $A(\tau)$, $N(t)$, and $S(t)$, respectively
s, $s(t)$ = Sounding (transmitted) signal form
S, $S(t)$ = Composite echo signal consisting of a linear superposition of weighted and delayed $s(t)$'s
$\mathcal{S}$ = Set of all possible $S(t)$ as determined by $\mathcal{A}$
T = Maximum possible target range (delay time)
X, $X(t)$ = Received datum = $S + N$
$\boldsymbol{\alpha}$ = A loss matrix with terms α_{ij} being the extra loss incurred for guessing i targets when there are really j targets
$\boldsymbol{\alpha}_i$ = A vector whose jth component is α_{ij}
$\delta(\tau)$ = Dirac delta function
$\lambda(\tau) = \int_0^T s(t)s(t + \tau)\, dt$, "autocorrelation function" of $s(t)$
$\hat{\boldsymbol{\lambda}}$ = A matrix consisting of terms $\hat{\lambda}_{ij} = \lambda(\hat{\tau}_i - \hat{\tau}_j)$
$\hat{\boldsymbol{\lambda}}^{-1}$ = The inverse of $\hat{\boldsymbol{\lambda}}$
ϕ, $\phi(\tau)$ = Matched-filtered version of $E_X[S(t)]$
$\hat{\boldsymbol{\phi}}$ = A vector whose ith component is $\hat{\phi}_i = \phi(\hat{\tau}_i)$
τ_i = Range (delay) of the ith target

The symbol $\hat{}$ is used over a parameter to indicate an estimate of that parameter; for example, $\hat{A}_i$ is an estimate of A_i. The subscript b is used with an estimate to indicate a Bayes estimate; for example, $\hat{A}_{ib}$ is the Bayes estimate of A_i. These symbols are also used in conjunction with terms like ϕ and λ when they are evaluated using Bayes estimates. For example, $\hat{\phi}_{ib} = \phi(\hat{\tau}_{ib})$. The symbols $\langle\ \rangle$ are used to indicate the (unconditional) expectation of the enclosed quantity.

Acknowledgment

The work reported in this paper is the result of the author's studies on radar resolution problems at the Rome Air Development Center. The author gratefully acknowledges the support and encouragement of his associates at the Center. Specific thanks are due to Lt. J. K. Wolf of the Rome Air Development Center for many illuminating suggestions. Also appreciated were the many stimulating discussions with Prof. N. Abramson of the Electrical Engineering Department of Stanford University.

Phase Shift Pulse Codes with Good Periodic Correlation Properties*

R. C. HEIMILLER†, MEMBER, IRE

Summary—A method of generating phase shift pulse codes of arbitrarily long length with zero periodic correlation except for the peak for zero shift is presented. The codes are of length p^2 where p is any prime number, and p different phase shifts corresponding to the pth roots of unity are necessary to generate them. Since p different phase shifts are required, these codes are not as easy to generate and process as the binary codes, but this does not seem to be a serious limitation to their usefulness. Application of these codes can be made as interpulse phase modulation for range resolution in pulse Doppler radars or for a method of synchronizing a pulse code communication system.

Introduction

BINARY codes[1] have been considered for modulating pulse trains at a carrier frequency for communication and radar systems. For example, a sequence $+1, +1, -1$ can correspond to either of the transmitted waveforms shown in Fig. 1. There is no reason to restrict the codes to $+1$ and -1 (*i.e.*, 0° and 180°) except for the added complexity in generating and processing for a number of different phase shifts.

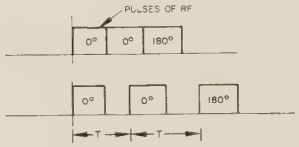

Fig. 1—Modulation waveforms for the code sequence $+1, +1, -1$.

Both single or repeated code groups can be considered. In a radar application, for example, a single code group can be used for pulse compression, or the entire time of observation of the target may be occupied by one group. On the other hand, many repetitions of a code group may occur during the time of observation, and therefore, the behavior of these repeated codes is of interest. For example, in a pulse Doppler radar, a code group may occupy an unambiguous range interval and be periodic with this interval. If this signal is then processed by a filter matched to a single code group and Doppler shifts are not considered, then the time or range response of these repeated codes is given by the periodic correlation function which is discussed in the next section. In the case of a search radar, there will be "end effects" due to the finite observation time or to the antenna scan pattern, but these effects will be small if a large number of code groups occur during the observation time.

Periodic Correlation

A modulation code of length n can be expressed as a sequence $\{a_0, a_1, a_2, \cdots, a_{n-1}\}$. The periodic (also called serial or cyclic) correlation function is defined as the sequence $\{x_0, x_1, x_2, \cdots, x_{n-1}\}$ where

$$x_i = \sum_{k=0}^{n-1} a_{k+i} a_k^*.$$

Note that $a_{n+m} = a_m$, and the asterisk denotes the complex conjugate. The modulation is expressed in complex form (*i.e.*, a phase shift of ϕ radians is written $e^{+j\phi}$). For $i = 0$, the value of x_i assumes its maximum:

$$x_0 = \sum_{k=0}^{n-1} |\, a_k \,|^2.$$

For $0 < i \leq n - 1$, the values of x_i should be low; in fact, zero if possible.

Note that this definition of the correlation function gives the values only for integral shifts. If the pulses form a contiguous signal, then the value for an intermediate shift $i + \delta$, $0 < \delta < 1$ is:

$$x_{i+\delta} = (1 - \delta) \sum_{k=0}^{n-1} a_{k+i} a_k^* + \delta \sum_{k=0}^{n-1} a_{k+i+1} a_k^*$$

$$= (1 - \delta) x_i + \delta x_{i+1}.$$

If the values of $x_i = 0$ for $0 < i \leq n - 1$, then the correlation function looks like that shown in Fig. 2.

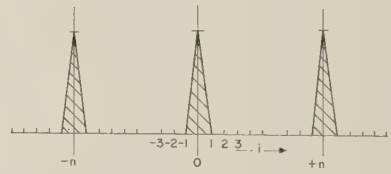

Fig. 2—Periodic correlation function for code group of length n with zero correlation except for peaks.

Elspas[2] has considered the generation and properties of binary codes (*i.e.*, $a = \pm 1$) with the specified periodic correlation function $\{n, -1, -1, \cdots, -1\}$, and he

* Received by the PGIT, January 11, 1961; revised manuscript received, June 9, 1961.

† Aerospace Group, Hughes Aircraft Co., Culver City, Calif.

[1] Binary codes consist of sequences of plus and minus ones. $+1$ corresponds to 0° phase shift of the carrier wave and -1 to 180° phase shift. Some examples, as well as a discussion of means of generation and processing are given by G. L. Turin, "An introduction to matched filters," IRE TRANS. ON INFORMATION THEORY, vol. IT-6, pp. 311–329; June, 1960.

[2] B. Elspas, "A Radar Based on Statistical Estimation and Resolution Considerations," Electronics Labs., Stanford University, Stanford, Calif., Tech. Rept. No. 361-1; August 1, 1955.

tabulates some of these codes up to length 127. Tompkins[3] has derived some properties of ternary codes (*i.e.*, $a = 1$, 0, or -1) with $x_i = 0$ for $0 < i \leq n - 1$, and he tabulates these codes up to length 18. In these studies, applications to radar systems are considered and the effects of Doppler shifts are investigated through the use of the "ambiguity" function.

Method of Generating[4] the Codes

Codes which have the value of $x_i = 0$ for $0 < i \leq n - 1$ can be generated in the following way. A prime number p is chosen greater than one. There are then p different pth roots of unity 1, ξ_1, ξ_2, $\cdots$, ξ_{p-1} (*i.e.*, roots of the form $e^{-j(2\pi k/p)}$ for $0 \leq k \leq p - 1$). Sequences of length p are formed as follows:

$$\begin{array}{llll} 1, & 1, & 1, & \cdots, 1 \\ 1, & \xi_1, & \xi_1^2, & \cdots, \xi_1^{p-1} \\ 1, & \xi_2, & \xi_2^2, & \cdots, \xi_2^{p-1} \\ \vdots & & & \vdots \\ 1, & \xi_{p-1}, & \xi_{p-1}^2, & \cdots, \xi_{p-1}^{p-1}. \end{array}$$

The code sequence is then formed by putting down in order all the first terms of each sequence above, then the second terms, then the third, etc. The resulting sequence of length p^2 is then:

$$1, 1, 1, \cdots, 1, 1, \xi_1, \xi_2, \xi_3 \cdots, \xi_{p-1}, 1, \xi_1^2, \xi_2^2 \cdots, \xi_{p-1}^2,$$
$$1, \xi_1^3 \cdots, 1, \xi_1^{p-1}, \xi_2^{p-1}, \cdots, \xi_{p-1}^{p-1}.$$

In Appendix II it is proved that this sequence has zero periodic correlation except for the peaks at $i = 0$, p^2, $2p^2$, etc. The order of the p sequences from which the final sequence is formed can be changed, and also any cyclic permutation of each sequence can be substituted for each sequence before constructing the code sequence without altering this result.

A larger class of similar sequences of the pth roots of unity of length $m = p^n - 1$ where n is a positive integer and p is prime have been described by Zierler.[5] These sequences have the specified periodic correlation function $\{m, -1, -1, \cdots, -1\}$ and can be generated by a linear shift register of n stages with each stage having p states.

Example: $p = 3$

The cube roots of unity are

$$1, e^{+j(2\pi/3)}, e^{-j(2\pi/3)}.$$

The preliminary sequences can be

$$\begin{array}{l} 1, 1, 1 \\ 1, e^{+j(2\pi/3)}, e^{-j(2\pi/3)} \\ 1, e^{-j(2\pi/3)}, e^{+j(2\pi/3)}. \end{array}$$

The code sequence is

$$1, 1, 1, 1, e^{+j(2\pi/3)}, e^{-j(2\pi/3)}, 1, e^{-j(2\pi/3)}, e^{+j(2\pi/3)}.$$

This sequence also has good nonperiodic correlation properties and is one of the sequences of the cube roots of unity which were found by DeLong and which have "optimum" nonperiodic correlation functions.[6]

The preliminary sequences can also be

$$\begin{array}{l} 1, 1, 1 \\ e^{+j(2\pi/3)}, e^{-j(2\pi/3)}, 1 \\ e^{+j(2\pi/3)}, 1, e^{-j(2\pi/3)}. \end{array}$$

In this case, the code sequence is

$$1, e^{+j(2\pi/3)}, e^{+j(2\pi/3)}, 1, e^{-j(2\pi/3)}, 1, 1, 1, e^{-j(2\pi/3)}.$$

The patient reader can verify that these perform as advertised.

Conclusions

Codes of arbitrarily long length with zero periodic correlation except for the peak can be generated by the above method in not much more time than it takes to write down the sequence. These codes exist for every length $n = p^2$ where p is a prime number. Moreover, many essentially different codes exist for each such n. p different phase shifts are required rather than just two as in the binary code case.

Further work can be done to determine how many essentially different codes exist for each prime number. Also, the "ambiguity" function of these codes should be investigated to determine their utility as a modulation for a pulse Doppler radar. For synchronization in a communication system, two code groups can be transmitted in a row and the received signal correlated with a single group. Synchronization pulses will then be generated as shown in Fig. 3.

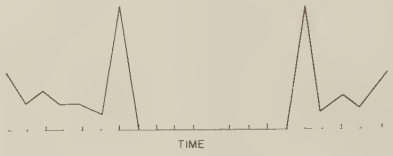

Fig. 3—Time synchronization pulses for communication system.

Appendix I

Notation and Preliminary Remarks in the Derivation

The expression

$$A = \{a_0, a_1, \cdots, a_{n-1}\}$$

means A is the sequence a_0, a_1, $\cdots$ a_{n-1}. If $B = \{b_0, b_1, \cdots b_{n-1}\}$, then $A \times B$ stands for the periodic cross

[3] D. N. Tompkins, "Codes with Zero Correlation," Engrg. Div., Hughes Aircraft Co., Culver City, Calif., Tech. Memo. 651; June, 1960.

[4] Generating in the mathematical sense.

[5] N. Zierler, "Linear recurring sequences," *J. Soc. Ind. and Appl. Math.*, vol. 7, p. 45, example 2; March, 1959.

[6] D. F. DeLong, Jr., "Three-Phase Codes," M.I.T. Lincoln Lab., Lexington, Mass., Group Rept. 47:28; July 24, 1959.

correlation function of A and B expressed as a sequence of length n. That is:

$$A \times B = \{x_0, x_1, \cdots, x_{n-1}\}$$

where

$$x_i = \sum_{k=0}^{n-1} a_{k+i} b_k \quad \text{where} \quad a_{n+m} = a_m.$$

Also

$$A + B = \{a_0 + b_0, a_1 + b_1, \cdots, a_{n-1} + b_{n-1}\}$$

and

$$A^* = \{a_0^*, a_1^*, \cdots, a_{n-1}^*\}$$

where a^* denotes the complex conjugate of a. The periodic correlation function of a single sequence A is defined as $A \times A^*$.

The Roots of Unity

The following remarks as well as Theorem I-1 are well-known results concerning the roots of unity.[7] The hth roots of unity in the complex number field are

$$1, e^{j(2\pi/h)}, e^{j2(2\pi/h)}, \cdots, e^{j(h-1)2\pi/h}.$$

The hth roots of unity form a cyclic group under multiplication. There are h distinct hth roots of unity. If h is a prime number, then all the roots except $+1$ are primitive roots. The following theorem is used extensively in the derivation.

Theorem I–1

If ξ is a hth root of unity, then:

$$1 + \xi + \xi^2 + \xi^3 + \cdots + \xi^{h-1} = h \quad \text{if} \quad \xi = 1$$
$$= 0 \quad \text{if} \quad \xi \neq 1.$$

Also note that $\xi^{h+i} = \xi^i$.

Appendix II

Derivation of Results

Let p be any prime number different from one. The pth roots of unity are denoted by $\xi_0, \xi_1, \cdots \xi_i, \cdots, \xi_{p-1}$ (no particular ordering of the ξ's is implied). Consider sequences of the form

$$C_i = \{1, \xi_i, \xi_i^2, \cdots, \xi_i^{p-1}\}.$$

Theorem II–1

$$C_n \times C_m = \{0, 0, 0, \cdots, 0\} \quad \text{if} \quad \xi_m \neq \xi_n^{-1}$$
$$= \{p, p\xi_n, p\xi_n^2, \cdots, p\xi_n^{p-1}\} \quad \text{if} \quad \xi_m = \xi_n^{-1}.$$

(Note that $\xi_n^{-1} = \xi_n^*$ since the ξ_n's are roots of unity in the complex number field. Therefore, $\xi_m \neq \xi_n^{-1}$ if and only if $C_m \neq C_n^*$, and $\xi_m = \xi_n^{-1}$ if and only if $C_m = C_n^*$.)

Proof

$$C_n \times C_m = \{x_0, x_1, x_2, \cdots, x_{p-1}\}$$

where

$$x_i = \sum_{k=0}^{p-1} \xi_n^{k+i} \xi_m^k$$
$$= \xi_n^i \sum_{k=0}^{p-1} \xi_n^k \xi_m^k$$
$$= \xi_n^i \sum_{k=0}^{p-1} (\xi_n \xi_m)^k.$$

If $\xi_m = \xi_n^{-1}$, then $\xi_n \xi_m = 1$ and

$$x_i = p\xi_n^i.$$

If $\xi_m \neq \xi_n^{-1}$, then $\xi_n \xi_m = \xi_q \neq 1$ and

$$x_i = \xi_n^i \sum_{k=0}^{p-1} \xi_q^k = 0$$

from Theorem I-1.

Corollary to Theorem II–1

Let D_i be any cyclic permutation of C_i. Then $D_n \times D_n^* = C_n \times C_n^* =$

$$\{p, p\xi_n, p\xi_n^2, \cdots, p\xi_n^{p-1}\}$$

and $D_n \times D_m^* = \{0, 0, \cdots, 0\}$ for $m \neq n$.

Theorem II–2

If u is an integer, $0 < u \leq p - 1$; then

$$\sum_{k=0}^{p-1} \xi_k^u = 0.$$

Proof

If $\xi_q \neq \xi_m$, then at least one of the ξ's is different from 1. Say $\xi_q \neq 1$. Then $\xi_m = \xi_q^v$ for some integer v, $1 < v \leq p - 1$ or $v = 0$.

$$\xi_m^u = (\xi_q^v)^u = (\xi_q^u)^v.$$

Since the ξ's are the pth roots of unity where p is prime, ξ_q^u is a primitive root of unity, call it ξ_n. Then $\xi_n \neq \xi_n^v$, since a primitive pth root of unity raised to the vth power $(1 < v \leq p - 1)$ generates a different primitive root of unity and $\xi_n^0 = 1 \neq \xi_n$. Therefore:

$$\xi_q^u \neq (\xi_q^u)^v = (\xi_q^v)^u = \xi_m^u.$$

Therefore, all the ξ_k^u's are different pth roots of unity since all the ξ_k's are different pth roots of unity. The sum of all the pth roots of unity is zero from Theorem I-1.

Now that these preliminary results have been proved, the proof of the main result can proceed. That is to prove that the codes generated by the methods outlined previously do have zero periodic correlation except for the

[7] *E. g.*, B. L. van der Waerden, "Modern Algebra," Frederick Ungar Publishing Co., New York, N. Y., vol. I, pp. 111–115; 1953.

main peaks. The code sequence of length p^2 is formed by interleaving p sequences of length p, which are arbitrary cyclic permutations of the C_i's defined at the beginning of this Appendix. The code sequence is

$$D = \{y_0, y_1, y_2, \cdots, y_{p^2-1}\}$$

and the sequences D_i from which it is formed are:

$$D_0 = \{y_0, y_p, y_{2p}, \cdots, y_{p^2-p}\}$$

$$D_1 = \{y_1, y_{p+1}, y_{2p+1}, \cdots, y_{p^2-p+1}\}$$

$$\vdots$$

$$D_{p-1} = \{y_{p-1}, y_{2p-1}, y_{3p-1}, \cdots, y_{p^2-1}\}.$$

All the members y_i of sequence D are defined from the fact that each D_i is a specified cyclic permutation of the corresponding C_i.

The correlation function of D can be expressed in terms of the auto- and cross-correlation functions of the sequences D_i.

$$D \times D^* = \{z_0, z_1, z_2, \cdots, z_{p^2-1}\}.$$

Consider the subsequences of length p:

$$G_i = \{z_i, z_{i+p}, z_{i+2p}, \cdots, z_{i+p^2-p}\}$$

$$\text{for } 0 < i \leq p-1$$

$$G_i = \sum_{m=0}^{p-1} D_{m-i} \times D_m^*$$

$$= \{0, 0, \cdots, 0\} \quad \text{if} \quad i \neq 0$$

$$G_0 = \sum_{m=0}^{p-1} \{p, p\xi_m, p\xi_m^2, \cdots, p\xi_m^{p-1}\}$$

from the corollary to Theorem II-1.

$$G_0 = \left\{p \sum_{m=0}^{p-1} 1, p \sum_{m=0}^{p-1} \xi_m, p \sum_{m=0}^{p-1} \xi_m^2, \cdots, p \sum_{m=0}^{p-1} \xi_m^{p-1}\right\}$$

$$= \{p^2, 0, 0, \cdots, 0\}$$

from Theorem II 2.

Therefore:

$$D \times D^* = \{p^2, 0, 0, \cdots, 0\}.$$

Optimum Nonlinear Filters for Quantized Inputs*

F. TUNG†, MEMBER, IRE, AND R. J. SCHWARZ‡, SENIOR MEMBER, IRE

Summary—Optimum least-square filters belonging to Zadeh's nonlinear class $\mathfrak{N}_1$ are considered. Attention is restricted to those systems whose present output is influenced only by a portion of the past input. The input signal consists of a message and noise, both of which are stationary random processes. It is assumed that the amplitude of the input time series is bounded and takes on discrete values at all times. This assumption leads to a nonlinear filter which can be realized as a quantizer or amplitude selector followed by a parallel set of linear filters. The system becomes optimum when the impulse responses of the linear filters satisfy a system of integral equations of the Wiener-Hopf type adapted to finite memory filters. By virtue of the assumptions made concerning the joint probability density functions of the message and noise processes, it is found that the Fourier transforms of the kernels of these equations are rational functions. A method is developed for the solution of this set of integral equations. This method is illustrated by an example, and the mean-square error of the nonlinear filter so obtained is compared with the best linear filter.

* Received by the PGIT, January 18, 1961. The work reported in this paper was supported by Contracts AF 19(604)4140 and Nonr 266(60).

† IBM Res. Lab., San José, Calif.; formerly Dept. of Elec. Engrg., Columbia University, New York, N. Y.

‡ Dept. of Elec. Engrg., Columbia University, New York, N. Y.

I. Introduction

SUBSEQUENT to the classic work of Wiener,[1] the subject of optimum linear filtering and prediction has been extended in many directions by Zadeh and Ragazzini[2] and many others. Inasmuch as a linear filter is a degenerate case of a nonlinear filter, improved results can usually be obtained by using a nonlinear filter. However, due to the incomplete knowledge of characterization of nonlinear systems as well as the inherent difficulty involved in an analytic treatment, relatively little work has been done with nonlinear filters.[3–5] In addition, we

[1] N. Wiener, "Extrapolation, Interpolation and Smoothing of Stationary Time Series," John Wiley and Sons, Inc., New York, N. Y.; 1950.

[2] L. A. Zadeh and J. R. Ragazzini, "An extension of Wiener's theory of prediction," *J. Appl. Phys.*, vol. 21, pp. 645–655; July, 1950.

[3] J. H. Laning, Jr., "Prediction and Filtering in the Presence of Gaussian Interference," Instrumentation Lab., Mass. Inst. Tech., Cambridge, Tech. Rept. R-27; October, 1951.

[4] R. Drenick, "A nonlinear prediction theory," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 146–162; September, 1954.

[5] L. A. Zadeh, "Optimum nonlinear filters," *J. Appl. Phys.*, vol. 24, pp. 396–404; April, 1953.

often find in practice that the amount of statistical data necessary for the design of nonlinear filters far exceeds what is available. This paper, therefore, considers a class of nonlinear systems whose optimization requires only second-order statistics.

The relation between the input $x(t)$ and the output $y(t)$ of the class of filters to be considered here is expressed by

$$y(t) = \int_0^T \bar{K}[x(t-\tau), \tau]\, d\tau \tag{1}$$

where T is the memory of the filter. Systems characterized by (1) have been designated by Zadeh[5] as class $\mathfrak{N}_1$. In his paper, Zadeh has derived a sequence of integral equations for a class of optimum filters and has shown that as the filter structure becomes more complicated, more and more information is necessary about the statistics of the input time series. It is found that only the second-order probability density functions of the message and the noise processes are necessary for obtaining the filter belonging to class $\mathfrak{N}_1$.

Since data supplied by computers and devices using digital readout are quantized, this information should be incorporated into the design of optimum filters. It is shown in this paper that this information can be used fruitfully by adopting the class of filters given by (1). The conventional mean-square error criterion is used.

Section II discusses the assumptions made which lead to the structure of the nonlinear filter in the form of a quantizer or amplitude selector followed by a parallel set of linear filters. A set of integral equations for the optimum impulse responses of these linear filters is obtained. The equations are of the Wiener-Hopf type, but are formulated for filters with a finite memory of T seconds. In Section III, a method is developed for solving these integral equations. Use of the method for infinite memory filters is also discussed. Section IV gives an example of pure prediction in which the method is applied. The improvement in mean-square error of the nonlinear filter over the best linear filter is then obtained.

II. Mathematical Formulation and the Structure of the Nonlinear Filter

Let the input signal $x(t)$ be the sum of two independent discrete-amplitude, wide-sense stationary random processes, namely, the message $m(t)$ and the undesired noise $n(t)$,

$$x(t) = m(t) + n(t). \tag{2}$$

The problem is to find a filter belonging to class $\mathfrak{N}_1$ such that the difference between the actual output from the filter and a desired output is minimized in some sense. Let $q[m(t+\alpha)](\alpha \geq 0)$ represent the desired output where $q(\cdot)$ is an odd function of its argument, and let $\epsilon(t)$ be the error between the desired output and the actual output $y(t)$,

$$\epsilon(t) = y(t) - q[m(t+\alpha)]. \tag{3}$$

As in Wiener's theory,[1] the least-mean-square error criterion will be used. For covenience, the ensemble average of $\epsilon(t)$ is assumed to be zero, so that the optimum predictor is that which minimizes the variance of $\epsilon(t)$. Let us denote the ensemble average by $\langle\ \rangle$. The filter is therefore optimum when

$$\langle \epsilon^2(t) \rangle = \text{minimum} \tag{4}$$

and

$$\langle \epsilon(t) \rangle = 0. \tag{5}$$

The following notation is used for convenience:

$x_1 = x(t-\tau_1)$
$x_2 = x(t-\tau_2)$
$m_\alpha = m(t+\alpha)$
$p_m(m_\alpha) = p_m[m(t+\alpha)]$
= first-order probability density function of $m(t+\alpha)$
$p_{xx}(x_1, x_2) = p_{xx}[x(t-\tau_1), x(t-\tau_2); \tau_1 - \tau_2]$
= joint probability density function of $x(t-\tau_1)$ and $x(t-\tau_2)$
$p_{xm}(x_1, m_\alpha) = p_{xm}[x(t-\tau_1), m(t+\alpha); \tau_1 + \alpha]$
= joint probability density function of $x(t-\tau_1)$ and $m(t+\alpha)$.

By the usual technique of variational calculus, we find[5] that a stationary solution occurs when the kernel $\bar{K}[x_2, \tau_2]$ satisfies the integral equation

$$\int_{-\infty}^{\infty} q[m_\alpha] p_{xm}(x_1, m_\alpha)\, dm_\alpha = \int_{-\infty}^{\infty} \int_0^T \bar{K}[x_2, \tau_2] p_{xx}(x_1, x_2)\, d\tau_2\, dx_2, \qquad 0 \leq \tau_1 \leq T. \tag{6}$$

It can be readily shown that this stationary solution does, in fact, give minimum mean-square error. The minimum mean-square error is given by

$$\text{Min}\ \langle \epsilon^2(t) \rangle = \langle (q[m_\alpha])^2 \rangle - \left\langle \left(\int_0^T \bar{K}[x_2, \tau_2]\, d\tau_2 \right)^2 \right\rangle. \tag{7}$$

Since $m(t)$ and $n(t)$ are discrete-amplitude processes, the joint probability density functions $p_{xx}(x_1, x_2)$ and $p_{xm}(x_1, m_\alpha)$ can be expressed in terms of delta functions. Let the message $m(t)$ have $2M$ amplitude levels symmetrical with respect to zero and designated by b_i, and let the input signal $x(t)$ have $2N$ amplitude levels symmetrical with respect to zero and designated by $c_i (N \geq M)$. Then

$$p_{xx}(x_1, x_2) = \sum_{i=-N}^{N} \sum_{j=-N}^{N} A_{i,j}(\tau_2 - \tau_1)\, \delta(x_1 - c_i)\, \delta(x_2 - c_j), \qquad i, j \neq 0 \quad (c_i = -c_{-i}) \tag{8}$$

and

$$p_{xm}(x_1, m_\alpha)$$

$$= \sum_{i=-N}^{N} \sum_{j=-M}^{M} B_{i,j}(\tau_1 + \alpha)\ \delta(x_1 - c_i)\ \delta(m_\alpha - b_j), \qquad (9)$$

$$i, j \neq 0 \qquad (b_j = -b_j)$$

where $\delta(\cdot)$ is the Dirac delta function. Further, since the input signal $x(t)$ can only assume $2N$ discrete values, it is seen that the kernel $\bar{K}$ in (1) depends, during any interval in which $x(t)$ assumes a value c_i, only on c_i and τ. Hence, $\bar{K}$ is completely specified by the $2N$ functions $\{K(c_i, \tau);\ j = \pm 1, \cdots, \pm N\}$. The nonlinear filter therefore can be represented by the structure shown in Fig. 1, where $K(c_i, \tau)/c_i$ are the unit impulse responses of linear filters. The system becomes linear when

$$\frac{K(c_j, \tau)}{c_j} = \frac{K(c_k, \tau)}{c_k} \quad \text{for all} \quad j \quad \text{and} \quad k. \qquad (10)$$

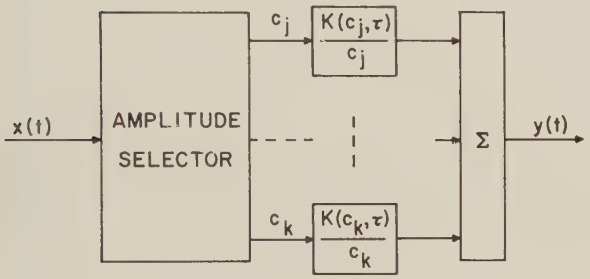

Fig. 1—Schematic representation of the nonlinear filter.

Substituting (8) and (9) into (6) and equating the corresponding coefficients of the delta functions associated with x_1, we find that the system becomes optimum when the set of functions $K(c_j, \tau_2)$ satisfies the following system of integral equations:

$$\sum_{j=-M}^{M} q(b_j) B_{i,j}(\tau_1 + \alpha)$$

$$= \sum_{\substack{j=-N \\ j\neq 0}}^{+N} \int_0^T A_{i,j}(\tau_2 - \tau_1) K(c_j, \tau_2)\, d\tau_2, \quad 0 \leq \tau_1 \leq T,$$

$$(i = \pm 1, \cdots, \pm N). \qquad (11)$$

For arbitrary $A_{i,j}(\tau_2 - \tau_1)$ and $B_{i,j}(\tau_1 + \alpha)$, the simultaneous set of integral equations in (11) is too formidable for any analytic solution. We shall now make two reasonable assumptions regarding the probability density functions so that a direct solution can be found.

1) The joint probability density functions p_{mm} of the message and p_{nn} of the noise are symmetrical with respect to their arguments as well as symmetrical with respect to the origin.

2) For all i and j, the quantities $[A_{i,j}(\tau_2 - \tau_1) - A_{i,-j}(\tau_2 - \tau_1)]$ can be approximated as the sums of a finite number of decaying exponentials. This is the logical extension of the usual assumption that the autocorrelation function, which is given by

$$R_x(\tau_2 - \tau_1) = \langle x_1 x_2 \rangle = \sum_i \sum_j c_i c_j A_{ij}(\tau_2 - \tau_1),$$

may be approximated by a sum of exponential functions.

We will now show that requiring the expected value of the error to be zero implies that

$$K(c_j, \tau) = -K(c_{-j}, \tau). \qquad (12)$$

From (5) and the fact that the ensemble mean of $q(\cdot)$ is zero for all t, we conclude that

$$\langle y(t) \rangle = 0; \qquad (13)$$

consequently

$$\int_0^T \langle \bar{K}[x(t - \tau), \tau] \rangle\, d\tau = 0. \qquad (14)$$

Let f_j denote the probability that $x(t - \tau)$ takes on the value c_j. Then in view of assumption 1),

$$f_j = f_{-j}, \qquad (j = 1, 2, \cdots, N). \qquad (15)$$

Eq. (14) becomes

$$\int_0^T \sum_{j=1}^{N} f_j [K(c_j, \tau) + K(c_{-j}, \tau)]\, d\tau = 0. \qquad (16)$$

In order for (16) to hold for any set of f_j satisfying $\sum_{j=1}^{N} 2f_j = 1$, it is necessary that

$$K(c_j, \tau) + K(c_{-j}, \tau) = 0 \qquad (j = 1, 2, \cdots, N) \qquad (17)$$

which establishes (12).

As a result of assumption 1), it is seen that $p_{xx}(x_1, x_2)$ is also symmetrical with respect to its arguments and with respect to the origin. On the other hand, $p_{xm}(x_1, m_\alpha)$ is only symmetrical about the origin. In addition, it is shown in Appendix I that $p_{xx}(x_1, x_2)$ is an even function in the variable $\tau_2 - \tau_1$. In terms of the coefficients in (8) and (9), these consequences of assumption 1) can be expressed as

$$A_{i,j}(|\ \tau_1 - \tau_2\ |) = A_{j,i}(|\ \tau_1 - \tau_2\ |)$$

$$= A_{-i,-j}(|\ \tau_1 - \tau_2\ |), \qquad (18)$$

$$B_{i,j}(\tau_1 + \alpha) = B_{-i,-j}(\tau_1 + \alpha). \qquad (19)$$

Let us define

$$z_i(\tau_1 + \alpha) \equiv \sum_{j=-M}^{M} B_{i,j}(\tau_1 + \alpha) q(b_j). \qquad (20)$$

Since $q(\cdot)$ is an odd function, it follows by using (19) that

$$z_i(\tau_1 + \alpha) = -z_{-i}(\tau_1 + \alpha), \qquad (i = 1, 2, \cdots, N). \qquad (21)$$

Eq. (11) can now be written

$$z_i(\tau_1 + \alpha) = \sum_{j=1}^{N} \int_0^T [A_{i,j}(|\ \tau_1 - \tau_2\ |)$$

$$- A_{i,-j}(|\ \tau_1 - \tau_2\ |)] \cdot K(c_j, \tau_2)\, d\tau_2,$$

$$0 \leq \tau_1 \leq T; \qquad (i = \pm 1, \pm 2, \cdots, \pm N). \qquad (22)$$

By using (18) and (21), it is seen that N of the $2N$ equations in (22) are actually redundant. Let us replace τ_1 and τ_2 by t and τ, respectively, and let

$$\omega_{i,j}(|\ t - \tau\ |) = A_{i,j}(|\ t - \tau\ |) - A_{i,-j}(|\ t - \tau\ |), \quad (23)$$

$$K_j(\tau) = K(c_j, \tau). \quad (24)$$

Eq. (22) finally reduces to

$$z_i(t + \alpha) = \sum_{j=1}^{N} \int_0^T \omega_{ij}(|\ t - \tau\ |)K_j(\tau)\, d\tau,$$

$$0 \leq t \leq T, \qquad (i = 1, 2, \cdots, N) \quad (25)$$

which is a system of integral equations of the Wiener-Hopf type, except that the upper limit of each integral is T instead of infinity. In addition, the Fourier transforms of the kernels are rational functions. From (7), we see that the minimum mean-square error is given by

$$\text{Min } \langle \epsilon^2(t) \rangle = \langle (q[m_\alpha])^2 \rangle - 2 \sum_{i=1}^{N} \int_0^T z_i(t + \alpha)K_i(t)\, dt. \quad (26)$$

III. Method of Solution

It was assumed that $\omega_{ij}(|\ \tau\ |)$ is the sum of a number of decaying exponential functions; hence its Fourier transform, defined as

$$W_{ij}(\lambda^2) = \int_{-\infty}^{\infty} \omega_{ij}(|\ \tau\ |)e^{-j\lambda\tau}\, d\tau, \quad (27)$$

is a rational function of λ^2. Suppose[6]

$$W_{ij}(\lambda^2) = \frac{D_{ij}P_{ij}(\lambda^2)}{Q(\lambda^2)} \quad (28)$$

where D_{ij} are constants, $Q(\lambda^2)$ is of order d and $P_{ij}(\lambda^2)$ is of order $n_{ij}(n_{ij} < d)$. We shall derive a necessary and sufficient condition under which an absolutely integrable solution of (25) exists, and then state the conditions under which the sufficient condition can be satisfied. $K_j(t)$ is absolutely integrable if

$$\int_0^T |\ K_j(t)\ |\, dt < \infty \qquad (j = 1, 2, \cdots, N) \quad (29)$$

which is the usual stability condition for linear systems. A system is, for our purposes, defined to be stable if all bounded inputs result in bounded outputs. The approach here is first to transform (25) to a simpler system of integral equations which can easily be analyzed. It is then shown that the solution of the modified system of integral equations does, in fact, satisfy (25). The following notation will be used:

$\hat{\omega}(|\ \tau\ |)$ = inverse Fourier transform of $1/Q(\lambda^2)$.

$[DP(d^2/dt^2)]$ = $N \times N$ square matrix whose elements are linear operators $D_{ij}P_{ij}(-d^2/dt^2)$.

$\mathbf{K}(t)$, $\mathbf{y}(t + \alpha)$, $\mathbf{z}(t + \alpha)$ = vectors whose components are time functions $K_j(t)$, $y_j(t + \alpha)$, $z_j(t + \alpha)$, respectively, $(j = 1, 2, \cdots, N)$.

Let $\mathbf{y}(t + \alpha)$ be any solution satisfying the following system of differential equations

$$\mathbf{z}(t + \alpha) = \left[DP\left(-\frac{d^2}{dt^2}\right)\right]\mathbf{y}(t + \alpha), \qquad 0 \leq t \leq T, \quad (30)$$

and let the modified set of integral equations be given by

$$\mathbf{y}(t + \alpha) = \int_0^T \hat{\omega}(|\ t - \tau\ |)\mathbf{K}(\tau)\, d\tau, \qquad 0 \leq t \leq T. \quad (31)$$

It is shown in Appendix II that the solution of (31) also satisfies (25). In fact, it can be shown[7] that all solutions of (25) are necessarily the solutions of (31). This latter property, however, is immaterial since any solution (if it is not unique) of (25) will give the same mean-square error.

We now investigate the solution of the modified system of integral equations (31). Appendix III shows that if $|\ \mathbf{K}(t)\ | < \infty$, $(0 \leq t \leq T)$, then $\mathbf{K}(t)$ satisfies

$$\mathbf{K}(t) = Q\left(-\frac{d^2}{dt^2}\right)\mathbf{y}(t + \alpha), \qquad 0 \leq t \leq T. \quad (32)$$

However, certain conditions on $\mathbf{y}(t + \alpha)$ at $t = 0$ and $t = T$ are necessary in order that $\mathbf{K}(t)$ so obtained from (32) do satisfy (31). Those conditions are obtained by substituting (32) into (31) and solving the resultant equations as an identity.[8]

Let us first establish a useful result. From the definition of $\hat{\omega}(|\ t\ |)$,

$$\hat{\omega}(|\ t\ |) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\lambda t}}{Q(\lambda^2)}\, d\lambda, \quad (33)$$

we obtain the corresponding differential equation

$$Q\left(-\frac{d^2}{dt^2}\right)\hat{\omega}(|\ t\ |) = \delta(t). \quad (34)$$

In particular, let

$$Q(\lambda^2) = \sum_{k=0}^{d} q_{2k}\lambda^{2k}. \quad (35)$$

Then (34) becomes[9]

$$\sum_{k=0}^{d} (-1)^k q_{2k}\hat{\omega}^{(2k)}(|\ t\ |) = \delta(t). \quad (36)$$

This relation will be used later on.

Using (35), (32) can be rewritten as

$$\mathbf{K}(t) = \sum_{k=0}^{d} (-1)^k q_{2k} \left(\frac{d}{d\tau}\right)^{2k} \mathbf{y}(t + \alpha), \qquad 0 \leq t \leq T. \quad (37)$$

[6] The common denominator of all $W_{ij}(\lambda^2)$ is used.

[7] F. Tung, "A Class of Optimum Nonlinear Filters for Quantized Inputs," Dept. of Elec. Engrg., Columbia University, New York, N. Y., Tech. Rept. T-2/N; July, 1960.

[8] W. B. Davenport, Jr., and W. L. Root, "An Introduction to the Theory of Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., Appendix II; 1958.

[9] $\hat{\omega}^{(2k)}(|t|)$ denotes $d^{2k}\hat{\omega}(u)/du^{2k}|_{u=|t|}$.

Multiplying both sides of (37) by $\hat{\omega}(|\ t - \tau\ |)$ and integrating the resultant expression with respect to τ from 0 to T, we obtain

$$\int_0^T \hat{\omega}(|\ t - \tau\ |)\mathbf{K}(\tau)\, d\tau = \int_0^t \sum_{k=0}^{d} (-1)^k q_{2k}\mathbf{y}^{(2k)}(\tau + \alpha)\hat{\omega}(t - \tau)\, d\tau + \int_t^T \sum_{k=0}^{d} (-1)^k q_{2k}\mathbf{y}^{(2k)}(\tau + \alpha)\hat{\omega}(\tau - t)\, d\tau, \qquad 0 \le t \le T. \tag{38}$$

Here we have separated the range of integration into two regions; for $0 \le \tau < t$ the kernel is $\hat{\omega}(t - \tau)$, while for $t \le \tau \le T$ the kernel is $\hat{\omega}(\tau - t)$. After integrating the right-hand side of (38) by parts $2d$ times and making use of the property that[10]

$$\hat{\omega}^{(i)}(0+) = (-1)^i\hat{\omega}^{(i)}(0-), \tag{39}$$

we are left with integrals

$$(-1)^k q_{2k}\int_0^T \mathbf{y}(\tau + \alpha)\omega^{(2k)}(|\ t - \tau\ |)\, d\tau$$

unintegrated as they occur at every other step. In addition, terms involving the derivatives of $\mathbf{y}(t + \alpha)$ at $t = 0$ and $t = T$ are carried over from each of the $2d$ integrations.

It can readily be shown[7] that the right-hand side of (39) can be expressed as

$$\sum_{e=1}^{2d} \hat{\omega}^{(e-1)}(t)\,\mathbf{Y}_e + \sum_{e=1}^{2d} \hat{\omega}^{(e-1)}(T - t)\mathbf{Z}_e + \int_0^T \sum_{k=0}^{d} (-1)^k q_{2k}\hat{\omega}^{(2k)}(|\ t - \tau\ |)\mathbf{y}(\tau + \alpha)\, d\tau,$$

where $\mathbf{Y}_e$ and $\mathbf{Z}_e$ are vectors whose components are Y_{ie} and $Z_{ie}(i = 1, 2, \cdots, N)$. Y_{ie} and Z_{ie} are linear combinations of the derivatives of $y_i(t + \alpha)$ at $t = 0$ and $t = T$ given by

$$Y_{ie} = -\sum_{k=k_e}^{d} (-1)^k q_{2k} y_i^{(2k-e)}(\alpha), \quad (i = 1, 2, \cdots, N), \tag{40}$$

and

$$Z_{ie} = \sum_{k=k_e}^{d} (-1)^{e+k+1} q_{2k} y_i^{(2k-e)}(T + \alpha), \qquad (i = 1, 2, \cdots, N), \tag{41}$$

where

$$k_e = \begin{cases} \dfrac{e+1}{2} & \text{if } e \text{ is odd} \\ e/2 & \text{if } e \text{ is even} \end{cases}.$$

[10] This can be verified by taking the limits of the derivatives of $\hat{\omega}(t)$ of any order on both sides of $t = 0$.

By (36), the summation in the last integral of the above expression is $\delta(t - \tau)$. Hence (38) can be reduced to

$$\int_0^T \hat{\omega}(|\ t - \tau\ |)\mathbf{K}(\tau)\, d\tau = \mathbf{y}(t + \alpha) + \sum_{e=1}^{2d} \hat{\omega}^{(e-1)}(t)\,\mathbf{Y}_e + \sum_{e=1}^{2d} \hat{\omega}^{(e-1)}(T - t)\mathbf{Z}_e, \qquad 0 \le t \le T. \tag{42}$$

We observe from (42) that, if the solution obtained from the differential equation (32) is to satisfy the system of integral equations (31), means should be provided to take care of the two terms added to $\mathbf{y}(t + \alpha)$ in (42).

Let us add to the solution $\mathbf{K}(t)$ of (32) two terms of the form

$$\mathbf{b}\ \delta(t) + \mathbf{c}\ \delta(t - T)^{11}$$

where $\mathbf{b}$ and $\mathbf{c}$ are vectors consisting of elements b_i, $c_i(j = 1, 2, \cdots, N)$. Then it is clear that (42) is identically satisfied if

$$\sum_{e=1}^{2d} \hat{\omega}^{(e-1)}(t)\,\mathbf{Y}_e = \hat{\omega}(t)\mathbf{b} \tag{43}$$

and

$$\sum_{e=1}^{2d} \hat{\omega}^{(e-1)}(T - t)\mathbf{Z}_e = \hat{\omega}(T - t)\mathbf{c}. \tag{44}$$

Let the roots of $Q(\lambda^2) = 0$ be denoted by $\pm\sqrt{-1}\ \alpha_i$, and let A_i be the residues of $1/Q(\lambda^2)$ at the poles $\lambda = \sqrt{-1}\ \alpha_i (i = 1, 2, \cdots, d)$. We see from (34) that $\hat{\omega}(t)$ can be expressed as

$$\hat{\omega}(t) = \sum_{i=1}^{d} A_i e^{-\alpha_i t}, \qquad t > 0.$$

It follows that

$$\hat{\omega}^{(e-1)}(t) = \sum_{i=1}^{d} (-\alpha_i)^{e-1} A_i e^{-\alpha_i t}, \qquad t > 0.$$

Upon equating the corresponding coefficients of $e^{-\alpha_i t}(i = 1, 2, \cdots, d)$ in (43) and (44), we obtain $2Nd$ algebraic equations. Therefore, the necessary and sufficient condition for an absolutely integrable solution of (25) to exist is that the $2Nd$ equations be identically satisfied.

A sufficient condition needed to obtain a solution of these $2Nd$ equations is that they contain $2Nd$ undetermined constants. Since $2N$ constants are contributed by $\mathbf{b}$ and $\mathbf{c}$, $2Nd - 2N = 2N(d - 1)$ constants must appear in $\mathbf{Y}_e$ and $\mathbf{Z}_e$. This requires, in view of (40) and (41), $2N(d - 1)$ constants in $\mathbf{y}(t + \alpha)$. Hence, the determinant $|\ DP(\lambda^2)\ |$ in (30) must be a polynomial in λ of degree $2N(d - 1)$. If the coefficient matrix of the $2Nd$ equations is nonsingular, the solution is then unique.

If some $n_{ij} = d$ and $|\ DP(\lambda^2)\ |$ is itself of order $2Nd$ in λ, then it can be shown that $\mathbf{b}$ and $\mathbf{c}$ are zero vectors, and hence the solution is bounded. It is of interest to note

[11] This does not violate (29), since $\int_{-\infty}^{\infty} |\ \delta(t)|\, dt = 1$.

that precisely these conditions arise if we consider the set of integral equations of the second kind

$$pK_i(t) = \sum_{j=1}^{N} \int_0^T \omega_{ij}(|\, t - \tau \,|)K_j(\tau)\, d\tau,$$
$$0 \le t \le T, \qquad (i = 1, 2, \cdots, N), \qquad (45a)$$

where p is an eigenvalue. These equations may be written

$$0 = \sum_{j=1}^{N} \int_0^T \hat{\omega}_{ij}(|\, t - \tau \,|)K_j(\tau)\, d\tau,$$
$$0 \le t \le T, \qquad (i = 1, 2, \cdots, N), \qquad (45b)$$

where

$$\hat{\omega}_{ij}(|\, t - \tau \,) = \begin{cases} -p\ \delta(t - \tau) + \omega_{ij}(|\, t - \tau \,|) & i = j \\ \omega_{ij}(|\, t - \tau \,|) & i \neq j. \end{cases}$$

The theory developed above immediately shows that (45b), and hence (45a), always possesses a solution if the Fourier transform of $\omega_{ij}(|\, x \,|)$ is a rational function in λ^2 expressed by (28) with $n_{ij} \le d$. The Fourier transform of $\omega_{ij}(|\, x \,|)$ is therefore also a rational function in λ^2 in which at least the degrees n_{ii} of the diagonal terms are equal to d. There are an infinite number of bounded solutions $\mathbf{K}(t)$. These solutions correspond to the set of eigenvalues p which make the coefficient matrix of the $2Nd$ equations singular.

So far, we have only considered the finite memory filter. The result, however, can be applied directly to the infinite memory filter, in which T is infinite. For the infinite memory filter, the stability condition becomes

$$\int_0^\infty |\, K_j(t) \,|\, dt < \infty, \qquad (j = 1, 2, \cdots, N)$$

which implies that:

1) The roots of
$$|\, DP(\lambda^2) \,| = 0 \qquad (46)$$
cannot be purely real.
2) The terms in $\mathbf{K}(t)$ which belong to the roots of (46) in the lower half of the λ plane should be discarded.

In the infinite memory case, it is necessary to use only (43). The number of equations as well as the number of unknowns are reduced by a factor of two.

IV. Example

For purposes of illustration, we shall consider here a simple example of pure prediction. Let the desired output be $m(t + \alpha)$. The amplitude of the input process at any time can take on any one of the four values ± 1 and ± 2 with equal probability. The second-order probability density function of the input process is assumed to be[12]

$$\begin{aligned} p_{mm}(m_1, m_2) = {} & \tfrac{1}{16}[1 - e^{-\beta_1|\tau_1-\tau_2|}][\delta(2, -2) \\ & + \delta(1, -1) + \delta(-2, 2) + \delta(-1, 1)] \\ & + \tfrac{1}{16}[1 - e^{-\beta_2|\tau_1-\tau_2|}][\delta(2, -1) \\ & + \delta(1, -2) + \delta(-2, 1) + \delta(-1, 2)] \\ & + \tfrac{1}{16}[1 - e^{-\beta_3|\tau_1-\tau_2|}][\delta(2, 1) \\ & + \delta(1, 2) + \delta(-2, -1) + \delta(-1, -2)] \\ & + \tfrac{1}{16}[1 + e^{-\beta_1|\tau_1-\tau_2|} + e^{-\beta_2|\tau_1-\tau_2|} + e^{-\beta_3|\tau_1-\tau_2|}] \\ & \cdot[\delta(2, 2) + \delta(1, 1) + \delta(-1, -1) + \delta(-2, -2)]. \end{aligned}$$

Let $\beta_1 = \beta_2 = 1$, $\beta_3 = 2$, and let $e^{-\alpha}$ be denoted by k. Using (8), (9), (20) and (23), we see that the optimizing integral equations (25) are

$$\begin{aligned} & \tfrac{1}{16}(5ke^{-t} - k^2e^{-2t}) \\ & \quad = \tfrac{1}{16}\int_0^T [3e^{-|t-\tau|} + e^{-2|t-\tau|}]K_1(\tau)\, d\tau \\ & \qquad + \tfrac{1}{16}\int_0^T [e^{-|t-\tau|} - e^{-2|t-\tau|}]K_2(\tau)\, d\tau \\ & \tfrac{1}{16}(7ke^{-t} + k^2e^{-2t}) \\ & \quad = \tfrac{1}{16}\int_0^T [e^{-|t-\tau|} - e^{-2|t-\tau|}]K_1(\tau)\, d\tau \\ & \qquad + \tfrac{1}{16}\int_0^T [3e^{-|t-\tau|} + e^{-2|t-\tau|}]K_2(\tau)\, d\tau, \\ & \qquad\qquad 0 \le t \le T. \end{aligned} \qquad (47)$$

The Fourier transforms of the respective kernels are

$$W_{11}(\lambda^2) = W_{22}(\lambda^2) = \frac{10(\lambda^2 + 2.8)}{\lambda^4 + 5\lambda^2 + 4}$$

and

$$W_{12}(\lambda^2) = W_{21}(\lambda^2) = \frac{2(-\lambda^2 + 2)}{\lambda^4 + 5\lambda^2 + 4}.$$

Eq. (30) can therefore be written as

$$\begin{bmatrix} 5ke^{-t} - k^2e^{-2t} \\ 7ke^{-t} + k^2e^{-2t} \end{bmatrix} = \begin{bmatrix} 10\left(-\frac{d^2}{dt^2} + 2.8\right) & 2\left(\frac{d^2}{dt^2} + 2\right) \\ 2\left(\frac{d^2}{dt^2} + 2\right) & 10\left(-\frac{d^2}{dt^2} + 2.8\right) \end{bmatrix} \mathbf{y}(t),$$
$$0 \le t \le T,$$

from which we obtain the solution

$$\begin{aligned} y_1(t) &= a_1e^{-\sqrt{2}t} + a_2e^{\sqrt{2}t} + a_3e^{2t} + a_4e^{-2t} + 1/6ke^{-t} \\ y_2(t) &= -a_1e^{-\sqrt{2}t} - a_2e^{\sqrt{2}t} + a_3e^{2t} + (a_4 - k^2/12)e^{-2t} \\ &\qquad + 1/3ke^{-t} \end{aligned}$$

[12] $\delta(i, j)$ denotes $\delta(m_1 - i)\delta(m_2 - j)$.

where a_1 to a_4 are arbitrary constants. $\mathbf{K}(t)$ and $\mathbf{y}(t)$ are related by (32) which, in this case, becomes

$$\mathbf{K}(t) = \left[\frac{d^4}{dt^4} - 5\frac{d^2}{dt^2} + 4\right]\mathbf{y}(t), \qquad 0 \leq t \leq T.$$

Upon simplification, we find that

$$\begin{aligned} K_1(t) &= g_1 e^{\sqrt{2}t} + g_2 e^{\sqrt{2}t} \\ K_2(t) &= -g_1 e^{-\sqrt{2}t} - g_2 e^{\sqrt{2}t} \end{aligned} \qquad 0 \leq t \leq T.$$

where g_1 and g_2 are arbitrary constants.

Now let us add $b_1\delta(t) + c_1\delta(t - T)$ and $b_2\delta(t) + c_2\delta(t - T)$ to $K_1(t)$ and $K_2(t)$, respectively, and substitute the complete expression in (47). We obtain, upon setting $T = 100$ msec,

$$\begin{aligned} g_1 &= 121.375 \times 10^{-3}(k^2 - k) \\ g_2 &= 2.775 \times 10^{-3}(k^2 - k) \\ b_1 &= (1291.9k - 291.9k^2) \times 10^{-3} \\ b_2 &= (1708.1k + 291.9k^2) \times 10^{-3} \\ c_1 &= 37.276 \times 10^{-3}(k^2 - k) \\ c_2 &= -c_1. \end{aligned}$$

The complete solution for $\mathbf{K}(t)$ is

$$\begin{aligned} K_1(t) &= g_1 e^{-\sqrt{2}t} - g_2 e^{\sqrt{2}t} \\ &\quad + b_1\,\delta(t) + c_1\,\delta(t - 0.1) \\ K_2(t) &= -g_1 e^{-\sqrt{2}t} - g_2 e^{\sqrt{2}t} \\ &\quad + b_2\,\delta(t) + c_2\,\delta(t - 0.1) \end{aligned} \qquad 0 \leq t \leq 0.1$$

$$\mathbf{K}(t) = 0 \qquad t > 0.1.$$

It is clear that $K_2(t) \neq 2K_1(t)$ (excluding the trivial case of $k = 1$). Consequently, the optimum predictor for this problem is always nonlinear. Using (26), we obtain the normalized mean-square error for this nonlinear filter,

$$\text{Min } \langle \epsilon^2 \rangle_{\mathfrak{N}_1} = 1 - \frac{1}{40}[37.0164k^2 + 1.9814k^3 + 1.0022k^4]. \qquad (48)$$

The normalized mean-square error of the corresponding linear filter is found to be

$$\text{Min } \langle \epsilon^2 \rangle_{\text{Linear}} = 1 - \frac{1}{40}[36.1046k^2 + 3.7920k^3 + 0.1034k^4]. \qquad (49)$$

It is seen that the difference between (49) and (48) is always greater than or equal to zero; that is

$$\text{Min } \langle \epsilon^2 \rangle_{\text{Linear}} - \text{Min } \langle \epsilon^2 \rangle_{\mathfrak{N}_1} = \frac{1}{40}[0.9118k^2 - 1.8106k^3 + 0.8988k^4] \geq 0.$$

Consequently, we have shown that, for this particular example, improved results can be obtained by using a nonlinear filter of class $\mathfrak{N}_1$ in place of a linear filter. The amount of improvement depends on the problem at hand. For the numerical values we have assumed, the improvement is negligible. For $k = 0.5$, the improvement is about 0.19 per cent. Inasmuch as there are five parameters in this example (β_1, β_2, β_3, k and T), it is unlikely that any general statement can be made with regard to their effects on the mean-square error without first obtaining an explicit expression in terms of these parameters.

For purposes of comparison, the case of the infinite memory filter was also examined. For $T = \infty$, it is found that

$$\begin{aligned} K_1(t) &= 121.3180 \times 10^{-3}(k^2 - k)e^{-\sqrt{2}t} \\ &\quad + (1292.898k - 292.898k^2) \times 10^{-3}\,\delta(t) \\ K_2(t) &= -121.3180 \times 10^{-3}(k^2 - k)e^{-\sqrt{2}t} \\ &\quad + (1708.102k + 292.898k^2) \times 10^{-3}\,\delta(t)' \end{aligned} \qquad 0 \leq t.$$

Table I shows the improvement of the class $\mathfrak{N}_1$ filter over the linear filter for two different values of k.

TABLE I

k	T	Improvement
0.5	0.1	0.19 per cent
0.5	∞	1.1 per cent
0.9	0.1	0.1 per cent
0.9	∞	2.6 per cent

Appendix I

We wish to show that the necessary and sufficient conditions for the joint probability distribution function

$$\text{Pr. } [x(t - \tau_1) \leq p_1;\, x(t - \tau_2) \leq p_2] \qquad (50)$$

to be an even function in $\tau_1 - \tau_2$ are:

1) $x(t)$ is a strictly stationary process of second order, and
2) The probability distribution function (50) is symmetrical with respect to the arguments p_1 and p_2, namely

$$\text{Pr. } [x(t - \tau_1) \leq p_1;\, x(t - \tau_2) \leq p_2] = \text{Pr. } [x(t - \tau_1) \leq p_2;\, x(t - \tau_2) \leq p_1]. \qquad (51)$$

Proof

Condition 1) implies that (50) is only a function of $\tau_1 - \tau_2$. Let $\beta = \tau_1 - \tau_2$; then (50) can be written as

$$\text{Pr. } [x(t) \leq p_1;\, x(t + \beta) \leq p_2]. \qquad (52)$$

Using (51), we see that (52) is equivalent to

$$\text{Pr. } [x(t) \leq p_2; x(t+\beta) \leq p_1]$$

which, in view of condition 1), can also be written as

$$\text{Pr. } [x(t-\beta) \leq p_2; x(t) \leq p_1],$$

or equivalently

$$\text{Pr. } [x(t) \leq p_1; x(t-\beta) \leq p_2]. \tag{53}$$

Sufficiency of the conditions is therefore proved by comparing (53) with (52). To prove their necessity, we we only need to proceed in the reverse direction. If (50) is to be a function of only $\tau_1 - \tau_2$, condition 1) is necessary. Let $\beta = \tau_1 - \tau_2$; then the statement that (50) is an even function in $\tau_1 - \tau_2$ implies that

$$\text{Pr. } [x(t) \leq p_1; x(t+\beta) \leq p_2] = \text{Pr. } [x(t) \leq p_1; x(t-\beta) \leq p_2]. \tag{54}$$

Using condition 1), we see that the right-hand side of (54) can be written as

$$\text{Pr. } [x(t+\beta) \leq p_1; x(t) \leq p_2] = \text{Pr. } [x(t) \leq p_2; x(t+\beta) \leq p_1]$$

which shows that condition 2) or (51) is satisfied.

Appendix II

We wish to show that the solution of (31) also satisfies (25). To see this, we operate on both sides of (31) by $[DP(-d^2/dt^2)]$. Hence

$$\left[DP\left(-\frac{d^2}{dt^2}\right)\right] \int_0^T \hat{\omega}(|\, t-\tau\,|)\mathbf{K}(\tau)\, d\tau = \left[DP\left(-\frac{d^2}{dt^2}\right)\right]\mathbf{y}(t+\alpha), \qquad 0 \leq t \leq T. \tag{55}$$

The left-hand side of (55) is a column matrix whose elements are

$$\sum_{j=1}^{N} D_{ij}P_{ij}\left(-\frac{d^2}{dt^2}\right) \int_0^T \hat{\omega}(|\, t-\tau\,|)K_j(\tau)\, d\tau, \qquad (i = 1, 2, \cdots, N),$$

or

$$\sum_{j=1}^{N} \int_0^T \omega_{ij}(|\, t-\tau\,|)K_j(\tau)\, d\tau, \qquad (i = 1, 2, \cdots, N). \tag{56}$$

In view of (30), the right-hand side of (55) is $\mathbf{z}(t+\alpha)$.

$$\mathbf{z}(t+\alpha).$$

Eq. (55) therefore reduces to

$$\sum_{j=1}^{N} \int_0^T \omega_{ij}(|\, t-\tau\,|)K_j(\tau)\, d\tau = z_i(t+\alpha), \qquad 0 \leq t \leq T, \qquad (i = 1, 2, \cdots, N),$$

which is, in fact, (25).

Appendix III

We wish to show that if $|\, \mathbf{K}(\tau)\,| < \infty$ $(0 \leq \tau \leq T)$, the solution of

$$\mathbf{y}(t+\alpha) = \int_0^T \hat{\omega}(|\, t-\tau\,|)\mathbf{K}(\tau)\, d\tau, \qquad 0 \leq t \leq T, \tag{57}$$

where

$$\hat{\omega}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{Q(\lambda^2)}\, e^{j\lambda x}\, d\lambda \tag{58}$$

satisfies

$$\mathbf{K}(t) = Q\left(-\frac{d^2}{dt^2}\right)\mathbf{y}(t+\alpha), \qquad 0 \leq t \leq T. \tag{59}$$

Since $\hat{\omega}(t)$ is of the form

$$\hat{\omega}(t) = \sum_{i=1}^{d} A_i e^{-\alpha_i |\, t\,|},$$

it is readily verified by taking the limits of the derivatives of any order on both sides of $t = 0$ that

$$\hat{\omega}^{(k)}(0+) = (-1)^k \hat{\omega}^{(k)}(0-). \tag{60}$$

Eq. (57) can be written

$$\mathbf{y}(t+\alpha) = \int_0^T \hat{\omega}(t-\tau)\mathbf{K}(\tau)\, d\tau + \int_t^T \hat{\omega}(\tau-t)\mathbf{K}(\tau)\, d\tau, \qquad 0 \leq t \leq T. \tag{61}$$

Taking derivatives on both sides with respect to t yields

$$\mathbf{y}^{(1)}(t+\alpha) = [\hat{\omega}(0+) - \hat{\omega}(0-)]\cdot\mathbf{K}(t) + \int_0^t \hat{\omega}^{(1)}(t-\tau)\mathbf{K}(\tau)\, d\tau - \int_t^T \hat{\omega}^{(1)}(\tau-t)\mathbf{K}(\tau)\, d\tau, \qquad 0 \leq t \leq T. \tag{62}$$

In view of (60), the first summation vanishes and we are left with

$$\mathbf{y}^{(1)}(t+\alpha) = \int_0^t \hat{\omega}^{(1)}(t-\tau)\mathbf{K}(\tau)\, d\tau - \int_t^T \hat{\omega}^{(1)}(\tau-t)\mathbf{K}(\tau)\, d\tau, \qquad 0 \leq t \leq T. \tag{63}$$

Similarly,

$$\mathbf{y}^{(2)}(t+\alpha) = [\hat{\omega}^{(1)}(0+) + \hat{\omega}^{(1)}(0-)]\mathbf{K}(t) + \int_0^t \hat{\omega}^{(2)}(t-\tau)\mathbf{K}(\tau)\, d\tau + \int_t^T \hat{\omega}^{(2)}(\tau-t)\mathbf{K}(\tau)\, d\tau, \qquad 0 \leq t \leq T. \tag{64}$$

The first term again vanishes and (64) is reduced to

$$\hat{\omega}^{(2)}(t+\alpha) = \int_0^t \hat{\omega}^{(2)}(t-\tau)\mathbf{K}(\tau)\,d\tau + \int_t^T \hat{\omega}^{(2)}(\tau-t)\mathbf{K}(\tau)\,d\tau, \qquad 0 \le t \le T. \tag{65}$$

Following the same reasoning, one sees that

$$\hat{\omega}^{(k)}(t+\alpha) = \int_0^t \hat{\omega}^{(k)}(t-\tau)\mathbf{K}(\tau)\,d\tau + (-1)^k \int_t^T \hat{\omega}^{(k)}(\tau-t)\mathbf{K}(\tau)\,d\tau, \qquad 0 \le t \le T. \tag{66}$$

For even k, we can write (66) as

$$\hat{\omega}^{(k)}(t+\alpha) = \int_0^T \hat{\omega}^{(k)}(|\,t-\tau\,|)\mathbf{K}(\tau)\,d\tau, \qquad 0 \le t \le T, \quad k = \text{even}. \tag{67}$$

From (58), we see that

$$\hat{\omega}^{(k)}(|\,x\,|) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(j\lambda)^k}{Q(\lambda^2)} e^{j\lambda x}\,d\lambda; \qquad k = \text{even}, \quad k \le 2d. \tag{68}$$

Consequently, if we operate on both sides of (57) by $Q(-d^2/dt^2)$, the denominator $Q(\lambda^2)$ vanishes and the resultant equation has the form

$$Q\left(-\frac{d^2}{dt^2}\right)\mathbf{y}(t+\alpha) = \int_0^T \mathbf{K}(\tau)\,d\tau \cdot \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{j\lambda(t-\tau)}\,d\lambda = \mathbf{K}(t), \qquad 0 \le t \le T,$$

where use has been made of the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\lambda(t-\tau)}\,d\lambda = \delta(t-\tau).$$

Correspondence

The Probability Density of the Phase Difference of a Narrow-Band Gaussian Noise with Sinusoidal Signal*

The first-order amplitude and phase[1] probability densities for a stationary narrow-band Gaussian noise both with and without a sinusoidal signal have been reported by several people.[2,3] The second-order probability density function of phase for noise alone was evaluated by MacDonald.[4] We have calculated the second-order probability function for phase with sinusoidal signal by approximation techniques and have numerically evaluated the probability density function for a phase difference between two instants of time for various parameter values. Values used are those applicable to a specific signal processing problem for which the work was done.[5] The numerical results, however, are presented in such a form that they may be of interest to workers in many fields to which statistical communication theory is applicable.

It is noted that this calculation is also applicable to the statistics of the outputs of a two-channel differencing device, where the inputs to the channel are identical sinusoidal signals with different Gaussian noises, the latter not necessarily being statistically independent.

Representation of the Signal-Plus-Noise Waveform

Consider a waveform $v(t)$ consisting of a sinusoidal signal $A\cos\omega_0 t + B\sin\omega_0 t$ superposed on a narrow-band stationary Gaussian noise $n(t)$ with zero mean, variance σ^2, bandwidth $\Delta\omega$, and center frequency ω_1.

Many writers[6] have represented such a noise by

$$n(t) = x_n(t)\cos\omega_1 t + y_n(t)\sin\omega_1 t \tag{1}$$

where $x_n(t)$ and $y_n(t)$ are random time functions varying slowly compared to $\cos\omega_1 t$ and $\sin\omega_1 t$. With this representation, the waveform $v(t)$ is given by

$$v(t) = x(t)\cos\omega_0 t + y(t)\sin\omega_0 t \tag{2}$$

where

$$x(t) = x_n(t)\cos(\omega_1-\omega_0)t + y_n(t)\sin(\omega_1-\omega_0)t + A$$

$$y(t) = -x_n(t)\sin(\omega_1-\omega_0)t + y_n(t)\cos(\omega_1-\omega_0)t + B$$

or alternatively by

$$v(t) = \rho(t)\cos(\omega_0 t - \phi(t)) \tag{3}$$

where

$$x(t) = \rho(t)\cos\phi(t)$$

$$y(t) = \rho(t)\sin\phi(t).$$

The probability density function of ϕ is illustrated in Fig. 1.

Summary of Calculation

The details of the calculation were presented earlier.[5] The procedure followed was:

1) The expression of the second-order statistics of $v(t)$ in terms of the quadrivariate Gaussian probability density function $p_2(\rho_1, \phi_1, \rho_2, \phi_2; \tau)$, where the subscripts 1 and 2 on ρ and ϕ refer to time instants t_1 and t_2, respectively, and τ is the separation time $(t_2 - t_1)$.
2) The approximate analytical integration of $p_2(\rho_1, \phi_1, \rho_2, \phi_2; \tau)$ over ρ_1 and ρ_2 to obtain the second-order phase probability density function $p_2(\phi_1, \phi_2; \tau)$.
3) The numerical integration of $p_2(\phi_1, \phi_1 + \Delta\phi; \tau)$ over ϕ_1 to obtain $p(\Delta\phi; \tau)$, the probability density function for the phase difference $\Delta\phi = \phi_2 - \phi_1$.

Results

The results of the numerical computations of $p(\Delta\phi, \tau)$ are contained in Tables I and II and in the curves of Figs. 2–5.[7] Fig. 6 is a plot of the cumulative distribution function, *i.e.*, the probability that $|\,\Delta\phi\,| \le \theta$ plotted

* Received by the PGIT, October 25, 1960; revised manuscript received, November 21, 1960.

[1] "Phase" throughout this note means "phase, modulo 2π." We do not concern ourselves with phase outside the interval $0 - 2\pi$.

[2] W. R. Bennett, "Methods of solving noise problems," Proc. IRE, vol. 44, pp. 609–638; May, 1956.

[3] W. B. Davenport, Jr., and W. L. Root, "An Introduction to the Theory of Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., sect. 8–6, pp. 165–167; 1958.

[4] D. K. C. MacDonald, "Some statistical properties of random noise," *Proc. Cambridge Phil. Soc.*, vol. 45, pp. 368 372; July, 1949.

[5] H. R. Raemer and R. Blyth, "Mathematical Model of Sonar Target Returns in Reverberation and Water Noise," Cook Res. Labs., Morton Grove, Ill., Phase I, Final Rept., Pt. II, Contract NObsr 87614, U. S. Navy Bur. of Ships; March 29, 1960.

[6] *E. g.*, Davenport and Root, *op. cit.*, sect. 8.5, p. 158.

[7] Figs. 2–5 are plots of probability density in reciprocal degrees vs $\Delta\phi$ in degrees, from $\Delta\phi = 0°$ to $\Delta\phi = 90°$. In all cases $p(\Delta\phi, \tau)$ continues to decrease out to 180° and then begins to increase. It is symmetrical about 180°.

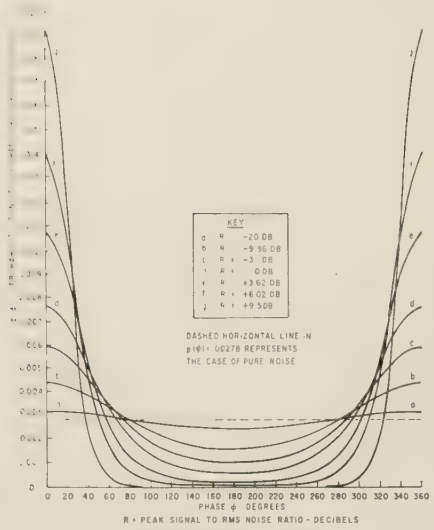

Fig. 1—First probability density function for phase.

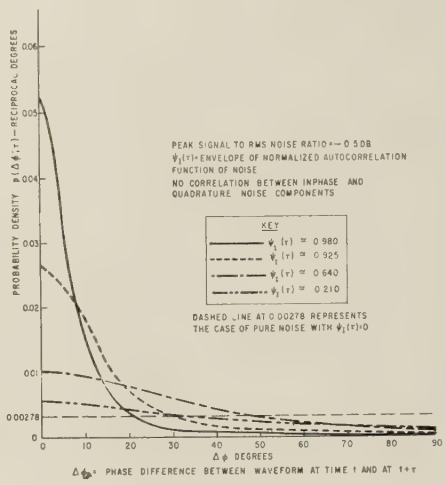

Fig. 2—Probability density $p(\Delta\phi; \tau)$ vs $\Delta\phi$.

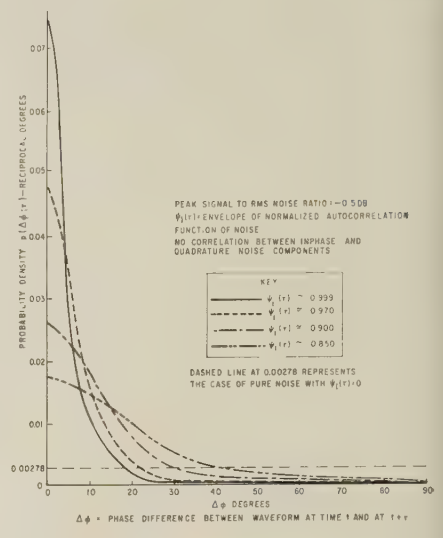

Fig. 3—Probability density $p(\Delta\phi; \tau)$ vs $\Delta\phi$.

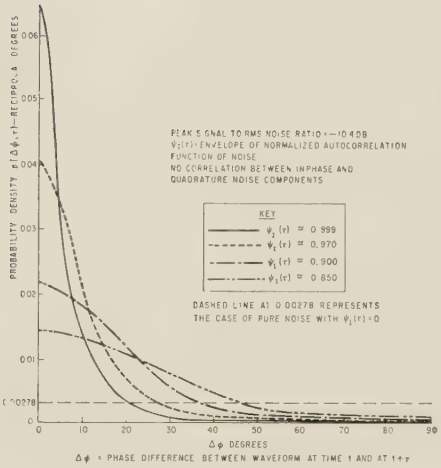

Fig. 4—Probability density $p(\Delta\phi; \tau)$ vs $\Delta\phi$.

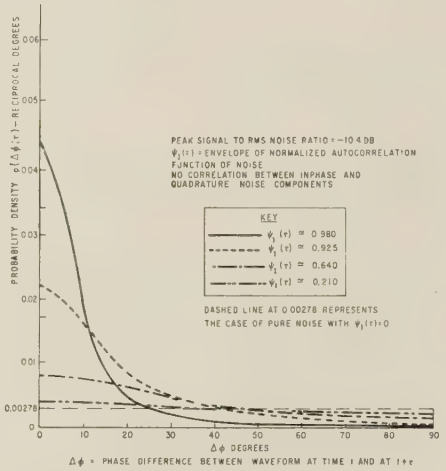

Fig. 5—Probability density $p(\Delta\phi; \tau)$ vs $\Delta\phi$.

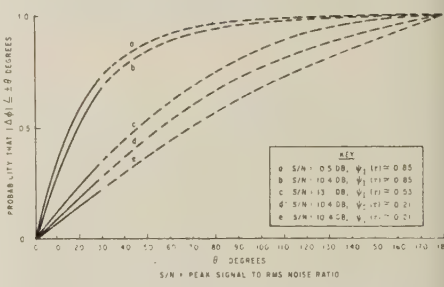

Fig. 6—Cumulative distribution of the phase difference, $\Delta\phi$.

TABLE I

TABULATION OF PROBABILITY DENSITY $p(\Delta\phi; \tau)$
$\psi_I(\tau)$ = ENVELOPE OF NORMALIZED AUTOCORRELATION FUNCTION OF NOISE
NO CORRELATION BETWEEN INPHASE AND QUADRATURE NOISE COMPONENTS

	Peak Signal / RMS Noise ≅ −0.5 db						Peak Signal / RMS Noise ≅ −10.4 db		
$\psi_I(\tau)$	$\Delta\phi = 0°$	$\Delta\phi = 3°$	$\Delta\phi = 10°$	$\Delta\phi = 20°$	$\Delta\phi = 40°$	$\Delta\phi = 90°$	$\Delta\phi = 0°$	$\Delta\phi = 3°$	$\Delta\phi = 90°$
0.999	4.43	3.25	0.667	0.116	0.163	0.0018	3.72	2.98	0.0028
0.970	2.87	2.47	0.955	0.228	0.357	0.0041	2.40	2.17	0.0065
0.900	1.54	1.45				0.0123	1.27	1.23	0.0203
0.850	1.030	0.988				0.0230	0.842	0.832	0.0396
0.770	0.805	0.778				0.0319	0.647	0.637	0.0568
0.650	0.618	0.600				0.0424	0.487	0.479	0.0785
0.460	0.459	0.446				0.0529	0.344	0.340	0.1033

TABLE II

TABULATION OF PROBABILITY DENSITY $p(\Delta\phi, \tau)$
$\psi_I(\tau)$ = ENVELOPE OF NORMALIZED CORRELATION FUNCTION OF NOISE
NO CORRELATION BETWEEN INPHASE AND QUADRATURE NOISE COMPONENTS

	Peak Signal / RMS Noise ≅ −10.4 db						Peak Signal / RMS Noise ≅ −0.5 db					
$\psi_I(\tau)$	$\Delta\phi = 0°$	$\Delta\phi = 3°$	$\Delta\phi = 10°$	$\Delta\phi = 20°$	$\Delta\phi = 40°$	$\Delta\phi = 90°$	$\Delta\phi = 0°$	$\Delta\phi = 3°$	$\Delta\phi = 10°$	$\Delta\phi = 20°$	$\Delta\phi = 40°$	$\Delta\phi = 90°$
0.980	2.61	2.32	1.113	0.281	0.476	0.0055	3.11	2.63	0.912	0.203	0.318	0.0035
0.925	1.29	1.24		0.506		0.020	1.56	1.46				0.0121
0.640	0.470	0.466				0.081	0.598	0.585				0.0432
0.210	0.230	0.227				0.121	0.330	0.320				0.0590
0.0775	0.189	0.189				0.123	0.286	0.276				0.0592
0.0655	0.187	0.185				0.123	0.283	0.272				0.0592
0.0464	0.182	0.180				0.123	0.276	0.269				0.0592

against θ. To render the numerical results applicable to situations other than the specialized problem whose parameters were used in the computer, they are presented for certain positive values of $\psi_I(\tau)$, the envelope of the normalized autocorrelation function of the noise waveform,[8] without reference to the physical parameter values on which basis these numbers were chosen.

Peak signal-to-rms noise ratios for which numerical work was done are $\simeq -0.5$ db (signal slightly below noise, Figs. 2 and 3) and $\simeq -10.4$ db (signal far below noise, Figs. 4 and 5).

Consider first the case where signal is slightly below noise. It is apparent from Figs. 2 and 3 that the probability density is sharply peaked at $\Delta\phi = 0$ if the noise waveforms at t_1 and $t_1 + \tau$ are highly correlated (*i.e.*, $\psi_I(\tau) \gtrsim 0.8$), and nearly flat when they are only slightly correlated (*i.e.*, $\psi_I(\tau) \lesssim 0.3$). The same applies qualitatively when the signal is far below noise, but as we would expect, the difference between high and low correlations is somewhat less pronounced, the waveform being almost pure noise and its phase being, therefore, more nearly random. The curves of Figs. 4 and 5 show that the density function flattens out even for relatively high correlations, *i.e.*, when $\psi_I(\tau) \approx 0.64$. Fig. 6 shows the probability that $\Delta\phi$ is within 10° of zero to be between 0.05 and 0.15 in cases where signal is well below noise and somewhat higher (between 0.25 and 0.35) when the signal is comparable to noise.

Note that the functional relationship with SNR of the probability density function of $\Delta\phi$ is similar to that of the first-order phase probability density $p(\phi)$ shown in Fig. 1. The latter function, however, uses the phase of the pure signal as the zero reference for the phase ϕ. The signal phase is subtracted out in $\Delta\phi$ and, therefore, the statistics of $\Delta\phi$ are independent of it. Thus, the use of $\Delta\phi$ in signal processing schemes would be advantageous over the use of ϕ, since the observer usually has no basis for the choice of a zero reference level for phase.

Several months after completion of the reported analysis, the authors learned of some Russian work by Tsvetnov[9] and Aleksandrov[10] that duplicates some of the results here. However, the calculation procedure was different. Both Tsvetnov and Aleksandrov calculated the probability density $p(\Delta\phi)$ directly by integrating out first ϕ_1 and then ρ_1 and ρ_2; and the authors obtained an approximate result for $p_2(\phi_1, \phi_2)$, expressed it in terms of ϕ_1 and $\Delta\phi$ and numerically integrated out ϕ_1. Also, the general calculation of $p_2(\phi_1, \phi_2)$ by the authors did not require the absence of correlation between inphase and quadrature components of the noise, whereas this specialization was made at the outset in the Russian work, and apparently it somewhat simplified the integration procedure.

Acknowledgment

The authors wish to express their appreciation to members of the Cook Technological Center staff for technical discussions and for the use of the Solid State 90 computer and other laboratory facilities. Specifically, we wish to acknowledge the help of A. Anderson, Technical Director of the Cook Research Laboratories, Dr. A. Gerlach, Technical Director of the Research Department, and J. Quinn, G. Lund, and J. Quinlan.

H. R. Raemer
Appl. Res. Lab.
Sylvania Electronic Systems
Waltham, Mass.
R. Blyth
Philco Corp.
Western Dev. Labs.
Palo Alto, Calif.
Both formerly at
Cook Res. Labs.
Morton Grove, Ill.

Prediction for Wide-Sense Markov Processes*

The minimum-mean-square linear estimator of $y(t) = \int_{-\infty}^{\infty} W(t, \tau)x(\tau)d\tau$ based on the present and past of the random process $x(t)$ is the solution of a generalized Wiener-Hopf equation. If $W(t, \tau)$ represents a nonrealizable time-varying weighting function and $x(t)$ is nonstationary, solving the Wiener-Hopf equation is difficult.[1] It is even more challenging if the estimate of $y(t)$ is predicated on a fragmentary portion of the past of $x(t)$, as when only sampled (not necessarily instantaneous or periodic) values of $x(t)$ are available.

As we shall show, the optimum estimator of $y(t)$ is readily determined if $x(t)$ is a Markov process in the wide sense. Indeed, for such processes we shall exhibit explicit solutions valid for $x(t)$ and $W(t, \tau)$ of the type discussed in the preceding paragraph. To describe a wide-sense Markov process,[2] we first define $\hat{E}$, the wide-sense conditional expectation. Let the random variable z and the random process $x(t)$ be of finite mean square. Then, for any arbitrary finite set of t's,

$$\hat{E}[z \mid x(t_1), x(t_2), \cdots, x(t_{n-1})] = \sum_0^{n-1} a_k x(t_k) \qquad (1)$$

with the a_k so chosen that the expectation $E[|\ z - \sum_0^{n-1} a_k x(t_k)\ |^2]$ is minimized.

A special case is obtained by taking $t_1 < t_2 < \cdots < t_n$ and $z = x(t_n)$. Then the process $x(t)$ is defined to be wide-sense Markov if any such set of t's yields $a_k = 0$ for $k = 1, 2, \cdots, n-2$; this is equivalent to

$$\hat{E}[x(t_n) \mid x(t_1), x(t_2), \cdots, x(t_{n-1})] = \hat{E}[x(t_n) \mid x(t_{n-1})] \qquad (2)$$

for all $t_1 < t_2 < \cdots < t_n$. Moreover, it may be verified that if $x(t)$ has a continuous correlation, and A is a subset of the real line such that $\sup_{\tau \epsilon A} \tau = \tau^* \leq t$,

$$\hat{E}[x(t) \mid x(\tau),\ \tau \,\varepsilon\, A] = \hat{E}[x(t) \mid x(\tau^*)] \qquad (3)$$

is implied by (2), and conversely.

We remark that $\hat{E}$ is a projection operator which has the special property (3) if $x(t)$ is wide-sense Markov. The value of $\hat{E}$ is easily computed when $\tau^* \leq t$. In fact, $x(\tau^*)/\{E[|\ x(\tau^*)\ |^2]\}^{1/2}$ is a one-member orthonormal family, so that $\hat{E}[x(t) \mid x(\tau^*)]$ is the one-term orthonormal expansion

$$\hat{E}[x(t) \mid x(\tau^*)] = R(t, \tau^*)x(\tau^*) \qquad (4)$$

where $R(s, t)$ is defined (somewhat unconventionally) as

$$R(s, t) = \frac{E[x(s)\overline{x(t)}]}{E[|\ x(t)\ |^2]}. \qquad (5)$$

Here the line over a quantity indicates its complex conjugate.

While the wide-sense Markov property is usually not directly verifiable, it is known that $x(t)$ has this property if and only if[3]

$$R(s,u) = R(s, t)R(t, u) \qquad s \leq t \leq u. \qquad (6)$$

If, in addition, $x(t)$ is wide-sense stationary, (6) requires that $E[x(s + t)\overline{x(s)}] = ke^{-c|t|}$, where $k \geq 0$ and c has a non-negative real part. Thus, wide-sense Markov processes are easily identified in practice.

If $x(t)$ is available to us for $t \,\varepsilon\, A$, the minimum-mean-square estimate of the previously defined $y(t)$ is given by[4]

$$\hat{y}(t) = \hat{E}[y(t) \mid x(\tau),\ \tau \,\varepsilon\, A]. \qquad (7)$$

The set A may be interpreted as needed for various practical applications. For example, the realizability restriction for the filter operating on $x(t)$ corresponds to $A = (-\infty, t]$. The additional constraint of finite memory filtering changes A to $[t - T, t]$, where T is the filter memory. A third alternative of importance is that of an $x(t)$ sampled before becoming available to a realizable filter; then A is the subset of $(-\infty, t]$ which contains only the sampling points[5] (or sampling intervals in the case of nonzero width samples).

We now find the optimum linear filter, when realizability is the only constraint. Substituting for $\hat{y}(t)$ in (7) and using the

[8] *I.e.*, $\psi_I(\tau) \cos \omega_1 = \overline{[n(t)n(t + \tau)]}/\overline{[n^2(t)]}$.

[9] V. V. Tsvetnov, "Statistical properties of signals and noise in two channel phase systems," *Radiotekh. i Elektron.*, vol. 12, no. 5, pp. 12–30; 1957. (In Russian.)

[10] M. S. Aleksandrov, "Distribution of changes in phase difference for fluctuating random signals and correlated random noise," *Radio Engrg. and Electronics* (Engl. transl. of *Radiotekh. i Elektron.*), vol. 5, pp. 360–365; 1960.

* Received by the PGIT, November 21, 1960; revised manuscript received, January, 1961. The research reported here was supported by NASA Research Grant NsG-2-59, and The University of Michigan Inst. of Science and Technology.

[1] See J. Laning and R. Battin, "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y., sect. 8.5, pp. 329 ff.; 1956.

[2] This concept and the resulting notation are due to J. Doob, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., pp. 77 and 90; 1953.

[3] *Ibid.*, p. 233. Here, as in (5), we take $R(s, t) = 0$ if $E[|\ x(t)\ |^2] = 0$.

[4] *Ibid.*, pp. 76–77.

[5] The sampled signal is properly viewed as a discrete sequence $\{x(t_n)\}$, where the t_n are the sampling times.

distributivity of $\hat{E}$ yields

$$\hat{y}(t) = \hat{E}\left[\int_{-\infty}^{t} W(t, \tau)x(\tau)\, d\tau \mid x(u), u \leq t\right] + \hat{E}\left[\int_{t}^{\infty} W(t, \tau)x(\tau)\, d\tau \mid x(u), u \leq t\right]. \quad (8)$$

Now, the operator $\hat{E}$ and the integration commute under conditions met by most $W(t, \tau)$.[6] In the first term of (8), we then have $\hat{E}[x(\tau) \mid x(u), u \leq t] = x(\tau)$, since $\tau \leq t$. In the second, we find that $\hat{E}[x(\tau) \mid x(u), u \leq t] = \hat{E}[x(\tau) \mid x(t)] = R(\tau, t)x(t)$, in accordance with (3) and (4). The result of our computation is therefore

$$\hat{y}(t) = \int_{-\infty}^{t} W(t, \tau)x(\tau)\, d\tau + x(t)\int_{t}^{\infty} W(t, \tau)R(\tau, t)\, d\tau \quad (9)$$

so that the optimum realizable filter $G(t, \tau)$ becomes

$$G(t, \tau) = W(t, \tau)U(t - \tau) + \left[\int_{t}^{\infty} W(t, v)R(v, t)\, dv\right]\delta(t - \tau). \quad (10)$$

Here, $U(\cdot)$ and $\delta(\cdot)$ are the usual unit step and delta functions, respectively.

Next, we solve the same problem as above, except that a finite memory filter is assumed. To accomplish this, we require the following assertion: if $x(t)$ is wide-sense Markov, and B is a subset of the real line such that $\inf_{\tau \varepsilon B} \tau = \tau_* \geq t$,

$$\hat{E}[x(t) \mid x(\tau), \tau \varepsilon B] = \hat{E}[x(t) \mid x(\tau_*)]. \quad (11)$$

The proof of (11) is as follows. In the first place, if $x(t)$ is wide-sense Markov, so is $x(-t)$; this is a consequence of (6). Applying (3) to $x(-t)$ gives $\hat{E}[x(-t) \mid x(-\tau), -\tau \epsilon - B] = \hat{E}[x(-t) \mid x((-\tau)^*)]$ whenever $(-\tau)^* = \sup_{\tau \varepsilon B}(-\tau) \leq -t$. Changing back to positive arguments yields the desired result, with $\tau_* = \inf_{\tau \varepsilon B} \tau = -\sup_{\tau \varepsilon B}(-\tau) = -(-\tau)^* \geq t$.

We now proceed as in the infinite memory case, except that (8) is replaced by

$$\hat{y}(t) = \hat{E}\left[\int_{-\infty}^{t-T} W(t, \tau)x(\tau)\, d\tau \mid x(u), t - T \leq u \leq t\right] + \hat{E}\left[\int_{t-T}^{t} W(t, \tau)x(\tau)\, d\tau \mid x(u), t - T \leq u \leq t\right] + \hat{E}\left[\int_{t}^{\infty} W(t, \tau)x(\tau)\, d\tau \mid x(u), t - T \leq u \leq t\right]. \quad (12)$$

The second and third terms of (12) are analogous to the two terms of (8), and are treated accordingly. The first term is simplified through use of (11). There results

$$\hat{y}(t) = \int_{t-T}^{t} W(t, \tau)x(\tau)\, d\tau + x(t - T)\int_{-\infty}^{t-T} W(t, \tau) \cdot R(\tau, t - T)\, d\tau + x(t) \cdot \int_{t}^{\infty} W(t, \tau)R(\tau, t)\, d\tau, \quad (13)$$

the corresponding weighting function being

$$G(t, \tau) = W(t, \tau)[U(t - \tau) - U(t - T - \tau)] + \left[\int_{-\infty}^{t-T} W(t, v)R(v, t - T)\, dv\right] \cdot \delta(t - T - \tau) + \left[\int_{t}^{\infty} W(t, v)R(v, t)\, dv\right] \cdot \delta(t - \tau). \quad (14)$$

The technique just presented is also useful in predicting $y(t)$ from a single sample. For instance, one instantaneous sample occurring at time t_0 makes $\hat{y}(t) = x(t_0)\int_{-\infty}^{\infty} W(t, \tau)R(\tau, t_0)d\tau$. A one-shot sample lasting from t_1 to t_2 will give a $y(t)$ similar to (13); we need only replace $t - T$ by t_1 throughout, and t by t_2 in all limits of integration and the $x(t)$ and $R(\tau, t)$ appearing in the third term of (13).

The optimum weighting function is now

$$G(t, \tau) = W(t, \tau)U(t - \tau) + \left[\int_{-\infty}^{t_1} W(t, v)R(v, t_1)\, dv\right]\delta(t_1 - \tau) + \left[\int_{t_2}^{\infty} W(t, v)R(v, t_2)\, dv\right]\delta(t_2 - \tau). \quad (15)$$

Note that the first term contains $U(t - \tau)$ rather than $U(t_2 - \tau) - U(t_1 - \tau)$, as might have been expected. This is purely a matter of convenience, since the filter receives no input before t_1 or after t_2, so that $\hat{y}(t)$ remains unchanged if $G(t, \tau)$ is arbitrarily specified for τ outside this interval.

Finally, we turn our attention to filtering a train of samples of $x(t)$. The methods which we have discussed here fail unless $W(t, \tau) = 0$ for $\tau < t$.[7] In spite of this restriction, it remains possible to deal with recovery of $x(t)$, prediction and/or differentiation—in fact, any totally nonrealizable operation on $x(t)$. Denoting the intersection of the sampling times and $(-\infty, t]$ by A, the assumption on $W(t, \tau)$ puts (7) in the form

$$\hat{y}(t) = \hat{E}\left[\int_{t}^{\infty} W(t, \tau) \cdot x(\tau)\, d\tau \mid x(u), u \varepsilon A\right]. \quad (16)$$

Now $u \leq \tau$ for all $u \varepsilon A$, so that $E[x(\tau) \mid x(u), u \varepsilon A] = R(\tau, \tau^*)$ from (3). Hence, we obtain

$$\hat{y}(t) = x(\tau^*) \cdot \int_{t}^{\infty} W(t, \tau)R(\tau, \tau^*)\, d\tau \quad (17)$$

where τ^* is the time of the trailing edge of the last sampling pulse occurring before time t (unless t is part of a sampling pulse, in which case $\tau^* = t$). It is clear from this result that only the trailing edge of the last sample is of importance; no useful contribution is obtained from the other (previous) pulses, regardless of their frequency or duration.

We shall apply the results of the preceding analysis to an example of prediction. First, however, let us establish the general solution for the optimum predictor. Prediction means that $y(t) = x(t + \alpha)$, where $\alpha \geq 0$ is the prediction interval. This $y(t)$ corresponds to $W(t, \tau) = \delta(t + \alpha - \tau)$. If the entire past of $x(t)$ can be used for prediction, substitution in (9) gives

$$\hat{x}(t + \alpha) = R(t + \alpha, t)x(t). \quad (18)$$

This states that an amplifier with time-varying gain $R(t + \alpha, t)$ is applied to $x(t)$ to give the best estimate, $\hat{x}(t + \alpha)$. It is clear that the same amplifier is also optimum with respect to any finite-memory class of linear filters.

When only sampled values of $x(t)$ are available, (17) is used, and

$$\hat{x}(t + \alpha) = R(t + \alpha, \tau^*)x(\tau^*) \quad (19)$$

where τ^* is the time of the trailing edge of the last sampling pulse occurring before

[6] A sufficient (but by no means necessary) condition for commutativity is as follows: the correlation function of $x(t)$ is continuous [as is already assumed for (3)], and $W(t, \tau)$ is absolutely integrable in τ for every t.

[7] Actually, it would suffice if $G(t, \tau)$ is zero for $\tau_* < \tau < \tau^*$. The computations are then simplified by use of both (3) and (11). Difficulties arise only when A and its complement are "interleaved."

time t (unless t is part of the sampling pulse, in which case $\tau^* = t$).

More specifically, consider the random process $x(t)$ defined for $t \geq 0$ by the differential equation

$$dx/dt + f(t)x = h(t)z(t)$$

$$x(0) = 0 \qquad (20)$$

in which $z(t)$ is a real random process with zero mean and covariance $E[z(s)z(t)] = \delta(s - t)$, $f(t)$ is any continuous function, and $h(t)$ is any measurable function.

Evidently, $x(t)$ is the output of a time-varying system whose input consists of white noise modulated by $h(t)$. For instance, $h(t)$ becomes a pulse train if the system receives its input through a sampling switch.

We establish that $x(t)$ is wide-sense Markov by verifying (6). For convenience, we define $g(u, v) = \exp\left[-\int_u^v f(\tau)\, d\tau\right]$ and note that $g(u, w) = g(u, v)g(v, w)$. A straightforward computation then yields

$$E[x(s)x(t)] = g(0, s)g(0, t) \cdot \int_0^{\min(s,t)} g^2(\tau, 0)h^2(\tau)\, d\tau = g(0, t) \cdot \int_0^{\min(s,t)} g(\tau, s)g(\tau, 0)h^2(\tau)\, d\tau \qquad (21)$$

so that for $s \leq t$,

$$R(s, t) = \frac{\int_0^s g(\tau, s)g(\tau, 0)h^2(\tau)\, d\tau}{\int_0^t g(\tau, t)g(\tau, 0)h^2(\tau)\, d\tau}. \qquad (22)$$

If the denominator of (22) is zero, we have the convention $R(s, t) = 0$. A direct substitution of (22) into (6) will now exhibit $x(t)$ as a wide-sense Markov process.

The optimum prediction of $x(t + \alpha)$ is obtained by substituting into (18). After simplifying the resulting expression, we see that the predictor is an amplifier with gain[8]

$$R(t + \alpha, t) = \exp\left[-\int_t^{t+\alpha} f(\tau)\, d\tau\right] \qquad (23)$$

which is calculated from (21). The result (23) holds only if $\int_0^t |\, h(\tau)\, |\, d\tau > 0$, for otherwise $\int_0^t g^2(\tau, 0)h^2(\tau)\, d\tau = 0$ implying $R(t + \alpha, t) = 0$; the latter gives $\hat{x}(t + \alpha) = 0$, as indeed it should.

If the prediction of $x(t + \alpha)$ is based only on samples of $x(t)$, (19) is applicable. Now $\hat{x}(t + \alpha) = x(\tau^*) \exp\left[-\int_{\tau^*}^{t+\alpha} f(\tau)\, d\tau\right]$ if $\int_0^{\tau^*} |\, h(\tau)\, |\, d\tau > 0$, and $\hat{x}(t + \alpha) = 0$ otherwise.

Frederick J. Beutler
College of Engrg.
The University of Michigan
Ann Arbor, Mich.

[8] This generalizes the simplest nontrivial example of Wiener prediction involving a wide-sense stationary process with correlation $e^{-c|t|}$, $c > 0$. Its solution is an amplifier with gain $e^{-c\alpha}$. The same result is obtained in our problem by specializing to $f(t) = c$, $h(t) = 1$ and starting the process at $t = -\infty$.

Optimum Prefiltering of Sampled Data*

Let f denote a stochastic signal and n denote additive noise; consider a prefilter which prepares $f + n$ for sampling. The sampled data is interpolated to recover an estimate of f. Optimum interpolation is discussed by Stewart,[1] and an aspect of optimum prefiltering is discussed by Spilker, Jr.[2]

Fig. 1 is a block diagram of the system.

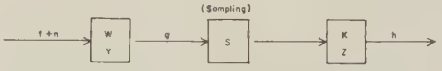

Fig. 1.

If the sample times are kT, then $h(t) = \sum_{k=-\infty}^{\infty} g(kT)\, K(t - kT)$ which is to be an estimate of f. For any prefilter frequency response Y, the optimum $Z(\omega)$, where Z is the Fourier transform of K, is found to be

$$Z(\omega) = T \frac{\bar{Y}(\omega)G_f(\omega)}{\sum_{k=-\infty}^{\infty}\left[G_f\left(\omega + \frac{2\pi k}{T}\right) + G_n\left(\omega + \frac{2\pi k}{T}\right)\right]\left|\, Y\left(\omega + \frac{2\pi k}{T}\right)\right|^2}$$

where G denotes power density spectrum. The optimization is in the least-square sense and numerous familiar assumptions are made, stationarity, etc. The corresponding mean-square error is

$$\overline{\epsilon^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_f(\omega) - \frac{G_f^2(\omega)\,|\, Y(\omega)\,|^2}{\sum_{k=-\infty}^{\infty}\left[G_f\left(\omega + \frac{2\pi k}{T}\right) + G_n\left(\omega + \frac{2\pi k}{T}\right)\right]\left|\, Y\left(\omega + \frac{2\pi k}{T}\right)\right|^2}\, d\omega.$$

The problem of interest here is to find the Y which minimizes this error. Suppose $G_n = 0$, then if no prefiltering is used ($Y(\omega) = 1$), there is no error at the sample times; however, it is proved below that this exactness at the sample times should always be sacrificed if minimum average-square error is to be obtained. Specifically, the optimum prefilter has a band-pass of $(T)^{-1}$ cps for any G_f and G_n.

We wish to maximize the integral of the second term in the expression for $\overline{\epsilon^2}$. Since its denominator is periodic with period $2\pi/T$, the quantity to be maximized can be written thusly:

$$\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \frac{\sum_{k=-\infty}^{\infty} G_f^2\left(\omega + \frac{2\pi k}{T}\right)\left|\, Y\left(\omega + \frac{2\pi k}{T}\right)\right|^2}{\sum_{k=-\infty}^{\infty}\left[G_f\left(\omega + \frac{2\pi k}{T}\right) + G_n\left(\omega + \frac{2\pi k}{T}\right)\right]\left|\, Y\left(\omega + \frac{2\pi k}{T}\right)\right|^2}\, d\omega.$$

* Received by the PGIT, December 5, 1960; revised manuscript received, February 6, 1961.

[1] R. M. Stewart, "Statistical design and evaluation of filters for the restoration of sampled data," Proc. IRE, vol. 44, pp. 253–257; February, 1956.

[2] J. J. Spilker, Jr., "Theoretical bounds on the performance of sampled data communications systems," IRE Trans. on Circuit Theory, vol. CT-7, pp. 335–341; September, 1960. In this paper, finite (positive) width samples are considered, and only an approximate analysis is made of optimum prefiltering.

We can consider any specific $|\omega| < \pi/T$ and now pick the values of $|Y(\omega + 2\pi k/T)|^2$ for all k to maximize the ratio

$$\frac{\sum_{k=-\infty}^{\infty} \frac{G_f^2\left(\omega + \frac{2\pi k}{T}\right)}{G_f\left(\omega + \frac{2\pi k}{T}\right) + G_n\left(\omega + \frac{2\pi k}{T}\right)} \theta_k}{\sum_{k=-\infty}^{\infty} \theta_k}$$

where θ_k has been defined as the kth term of the sum in the denominator. For the moment, let $\sum_{k=-\infty}^{\infty}\theta_k$ be a fixed positive number. To find the $|Y(\omega + 2\pi k/T)|^2$ is enough to find the θ_k's. The numerator is the dot product of a vector in the first hyper-quadrant wth the vector θ, each component of θ is non-negative, and for the moment we fix $\sum_{k=-\infty}^{\infty}\theta_k$. In two-dimensional space, the problem is to pick $\boldsymbol{\theta}$ having its tip on the line indicated in Fig. 2 so as to maximize its projection along

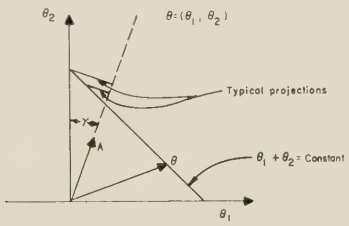

Fig. 2.

a fixed vector (which has been denoted by A). Clearly,[3] the best $\boldsymbol{\theta}$ is obtained by putting all of $\sum_{k=-\infty}^{\infty}\theta_k$ on the axis for which $\mathbf{A}$ has its largest component; when this is done, the ratio becomes

$$\frac{|A|\,|\boldsymbol{\theta}|\cos\gamma}{\sum_{-\infty}^{\infty}\theta_k}$$

but with the optimum $\boldsymbol{\theta}$, $|\boldsymbol{\theta}| = \sum\theta_k$ and hence the ratio we maximized has the value $|\mathbf{A}|\cos\gamma$. This is independent of the value we used for $\sum\theta_k$.

Hence the optimum values for $|Y(\omega + 2\pi k/T)|$ consist of all zeros except for that k for which

$$\left\{\frac{G_f^2\left(\omega + \frac{2\pi k}{T}\right)}{G_f\left(\omega + \frac{2\pi k}{T}\right) + G_n\left(\omega + \frac{2\pi k}{T}\right)}\right\}$$

is a maximum. If the maximum is achieved for several terms of the sequence, the values on these terms can be selected arbitrarily (not all zero) with all other terms zero. At the k which provides the maximum, the nonzero value of $|Y(\omega + 2\pi k/T)|$ is not important[4] (for mathmatical reasons the values used should be such that $|Y|$ is a reasonably well-behaved function). Suppose the value 1 is used for $|Y|$. The optimum $Y(\omega)$, of course, now consists of a function which is one on a set of measure $2\pi/T$ and zero elsewhere. The set where Y is one is determined by considering the ω-axis decomposed into a set of equivalence classes where $\omega_1 \overset{E}{=} \omega_2$ if and only if $\omega_2 - \omega_1 = 2\pi k/T$ for some integer k, *i.e.*, provided ω_1 and ω_2 are aliases. Then Y is zero on all but one point of each class, and that point is determined by the point (or a point) where $G_f^2/G_f + G_n$ has its maximum.

As an important special case, if $G_f^2/G_n + G_f$ (or simply G_f in the noise free case) is even and nonincreasing for positive frequencies, then the best $Y(\omega)$ is

$$Y(\omega) = \begin{cases} 1 & \text{for } |\omega| < \pi/T \\ 0 & \text{for } |\omega| > \pi/T \end{cases}.$$

W. M. Brown
Elec. Engrg. Dept. and
Inst. of Science and Technology
The University of Michigan
Ann Arbor, Mich.

[3] The fact that this observation generalizes is well known in the theory of linear programming.

[4] If the sampler is noisy, an additional noise is added at the output of the prefilter, and hence Y should be made large if the effect of this additional noise is to be made small.

Comments On a Paper By Wax*

In a recent article,[1] Wax determined upper bounds for the maximum number of code symbols in a Hamming[2] code of length n and distance D. The purpose of this note is to point out a simple argument which will result in an improvement on Wax's bound when D is small compared with n. For $D > n/2$, the argument leads to the same bound obtained by Wax.

Wax's argument, which depends on the density method of Blichfeldt,[3] is best expressed in physical terms: if it can be shown that the density of a body is everywhere ≤ 1, then it follows that the mass of the body is always less than or equal to its volume. Following Wax, we imagine that at each of the code points we have centered a sphere of radius $R = (D/2)^{1/2}$ where the Euclidean distance d corresponding to the Hamming distance D is given by $D^{1/2} = d$. These spheres which are thought of as superposable are truncated along the faces of the unit cube and assigned a density as indicated by Wax.

Letting $N(n, D)$ denote the maximum number of code points in a space of dimension n whose distance apart is at least D, and letting $M(n, D)$ denote the density function, Wax shows that

$$N(n, D)M(n, D) \leq 1$$

by use of the argument mentioned earlier. The argument consists of showing that even though these new concentric spheres overlap, the choice of the density function $M(n, D)$ is such that the density at any point is less than or equal to unity. This, together with the observation that the truncated spheres lie completely within the unit cube, lead to the bound.

An improvement in the bound can be obtained by a better estimate of the total volume W of the overlapping spheres centered at the code points. We imagine the cube to be centered at the origin so that the code points will always come from the set $(\pm 1/2 \cdots \pm 1/2)$. Suppose for a moment that we center a truncated sphere of radius $(D/2)^{1/2}$ at each vertex of the cube as though every vertex were a code point. There will, of course, be considerable overlapping, but the total volume Y of these 2^n truncated spheres satisfies $W \leq Y \leq 1$ so that Y bounds W. To calculate Y, the unit cube is subdivided into 2^n cubes of edge 1/2 denoted by $C_i^{1/2}$ $i = 1, 2 \cdots 2^n$. Associated with these smaller cubes are 3^n vertices, namely the points $(\pm 1/2 \cdots \pm 1/2)$. With each $C_i^{1/2}$ there is associated one and only one vertex X_i from the set $(\pm 1/2 \cdots \pm 1/2)$. Any point y which belongs to the unit cube must be in the interior or on the boundary of one of the $C_i^{1/2}$. It follows that y will be contained in some sphere if and only if $|y - Xi| \leq (D/2)^{1/2}$. In the event that one of the coordinates of y is zero, this coordinate is arbitrarily taken to be positive. It follows that the unit cube is covered by the 2^n spheres of radius $(D/2)^{1/2}$ centered at the vertices in the same way as the cube $C_i^{1/2}$ is covered by a single sphere centered at Xi. It is shown by Wax that the volume of the truncated sphere of radius R centered at the vertex of a cube of side b is given by $b^n V(n, R/b)$. Letting $b = 1/2$, we obtain

$$N(n, D)M(n, D) \leq V(n, (2D)^{1/2}).$$

This bound will grow better as D becomes small compared with n. For $D > n/2$, $V(n, (2D)^{1/2}) = 1$, so that in this case there is no improvement.

Louis D. Grey
The Teleregister Corp.
Stamford, Conn.

* Received by the PGIT, December 16, 1960.

[1] N. Wax, "On upper bounds for error detecting and error correcting codes of finite length," IRE Trans. on Information Theory, vol. IT-5, pp. 168–174; December, 1959.

[2] R. W. Hamming, "Error detecting and error correcting codes," *Bell Sys. Tech. J.*, vol. 29, pp. 147–160; April, 1950.

[3] H. F. Blichfeldt, "The minimum value of quadratic forms and the closest packing of spheres," *Math. Ann.*, vol. 101, pp. 605–608; 1929.

On the Postdetection Correlation Between Two Sinusoidal Signals with Superimposed Correlated Noise*

Introduction

The problem of measuring autocorrelation and cross-correlation is frequently encountered in experimental analysis. Correlation measurements are involved in the majority of experimental procedures in connection with analysis of transmission mechanisms, as, for instance, ionospheric transmission and scattering of radio waves. Another example is the method of finding the transfer function of linear four-poles by measuring the correlation between input and output signals. The results of the present paper have also been successfully applied to the analysis of signals from earth satellites.

Since most of the signals used in transmission techniques are high-frequency signals with relatively narrow bandwidths, the correlation measurements are most easily performed on the detected signals, whereas for theoretical calculations the correlation between the undetected signals is required. It is therefore of great importance to know the relationship between the correlation before and after detection.

It is the purpose of this paper to establish this relationship for a special class of signals, namely signals consisting of a sinusoidal component with superimposed narrow-band Gaussian noise. This class of signals is representative for many situations appearing in technical applications.

The mathematical methods for solving such problems are given by Rice[1] and Middleton.[2]

We consider two signals, $I_1(t)$ and $I_2(t)$, both consisting of a sinusoidal component with narrow-band Gaussian noise superimposed:

$$\begin{aligned} I_1(t) &= S_1(t) + N_1(t) \\ I_2(t) &= S_2(t) + N_2(t) \end{aligned} \tag{1}$$

where

$$\begin{aligned} S_1(t) &= P_1 \cos(2\pi f_0 t - \phi_1) \\ S_2(t) &= P_2 \cos(2\pi f_0 t - \phi_2). \end{aligned} \tag{2}$$

The noise signals $N_1(t)$ and $N_2(t)$ are supposed to be correlated, both having a narrow power spectrum with center frequency near f_0. The power spectra are $w_1(f)$ and $w_2(f)$, respectively. The SNR of the signal I_1 is given by

$$\alpha_1 = \frac{P_1^2}{2\psi_{01}} \tag{3}$$

where

$$\psi_{01} = \int_0^\infty w_1(f)\, df. \tag{4}$$

Similar expressions apply to the second signal. The analysis includes both cross-correlation and autocorrelation. In the latter case, I_2 is defined by

$$I_2(t) = I_1(t + \tau). \tag{5}$$

We introduce two quantities ρ_f and ϑ, defined by

$$\rho_f e^{i\vartheta} = \left\langle \frac{1}{\sqrt{\psi_{01}\psi_{02}}} \cdot \int_0^\infty \sqrt{w_1(f)w_2(f)}\, e^{i\theta(f)}\, df \right\rangle \tag{6}$$

where $\theta(f)$ is the phase difference between the components of the two noise signals at frequency f, and the pointed brackets $\langle\rangle$ denote the statistical average. With these notations, the correlation coefficient ρ_N of the two noise signals is given by

$$\rho_N = \rho_f \cos\vartheta. \tag{7}$$

We shall consider a linear envelope-detector consisting of a linear detector followed by a low-pass filter. The output is then proportional to the envelope of the input signal. We shall also consider a square-law detector followed by a low-pass filter. The output of this device is proportional to the square of the envelope of the input signal. Our task is therefore to calculate the correlation coefficient ρ_R for the two envelopes of the signals, and the correlation coefficient ρ_{R^2} for the squares of the envelopes.

Results of the Analysis

Explicit expressions for ρ_R and ρ_{R^2} may be obtained by the characteristic function method if the properties of the detectors are expressed by their contour integral representation. This method has been described in detail.[1,2]

We obtain for the square-law detector:

$$\rho_{R^2} = \frac{1}{1 + 2\sqrt{\alpha_1\alpha_2}} \cdot [\rho_f^2 + 2\sqrt{\alpha_1\alpha_2}\, \rho_f \cos\varphi] \tag{8}$$

where

$$\varphi = \phi_1 - \phi_2 - \vartheta. \tag{9}$$

Eq (8) contains four parameters: The correlation coefficient ρ_{R^2} of the detected signals; the "in-phase" correlation coefficient ρ_f of the undetected signals; the geometric mean of the SNR's $\sqrt{\alpha_1\alpha_2}$; and the phase factor φ. In order to determine the correlation between the undetected signals from the measured values of ρ_{R^2}, one has to know certain properties of the signals, namely the SNR's and the phase angle φ.

In Fig. 1, the correlation coefficient ρ_{R^2} of the detected signals is plotted vs the correlation coefficient ρ_f with $\sqrt{\alpha_1\alpha_2}$ and φ as parameters. Fig. 1 shows the general behavior of the function. In Fig. 2 is shown a more detailed plot for the special case that the phase angle φ is equal to zero. This is probably the case most often met with in practical applications. In the case of autocorrelation, it is easily shown that φ equals zero for symmetrical noise spectrum.

For the square-law detector, the problem leads to integrals which can be evaluated, giving a simple expression in closed form for the correlation between the detected signals. This is not the case for the linear detector, for which series expansion must be used.

The exact expression for the general case is very complicated and requires electronic computers for its evaluation. In Fig. 3 is shown a plot of ρ_R as a function of ρ_f for the special case that $\varphi = 0$ and $\alpha_1 = \alpha_2 = \alpha$.

Comparison with Fig. 2 shows that there is little difference between the correlation of the outputs from linear detectors and from square-law detectors.

The series expansion for ρ_R may be approximated by simpler expressions valid for certain ranges of the parameters. We shall state here only the results for the case of equal SNR's $\alpha_1 = \alpha_2 = \alpha$, and $\varphi = 0$. The following three approximate expressions are then obtained:

$$\rho_R \approx \frac{2}{(4 - \pi)(1 + \alpha)} \left\{ 2E(\rho_f) - (1 - \rho_f^2)K(\rho_f) - \frac{\pi}{2} + \alpha\left[\left(1 + \frac{1}{\rho_f}\right)E(\rho_f) - \frac{1 - \rho_f^2}{\rho_f} K(\rho_f) - \frac{\pi}{2} \right] \right\} \tag{10}$$

which is valid for small values of α, $\alpha \ll 1$;

$$\rho_R \approx \rho_f \frac{1 - \frac{1}{4\alpha}(2 - \rho_f)}{1 - \frac{1}{4\alpha}} \tag{11}$$

which is valid for large values of α, $\alpha \gg 1$; and

$$\rho_R \approx \rho_f \frac{\alpha\left[I_0\left(\frac{\alpha}{2}\right) + I_1\left(\frac{\alpha}{2}\right) \right]^2 + \frac{\rho_f}{2}\left[I_0\left(\frac{\alpha}{2}\right)^2 + I_1\left(\frac{\alpha}{2}\right) \right]^2}{8e^\alpha(1 + \alpha)/\pi - 2\left[(1 + \alpha) I_0\left(\frac{\alpha}{2}\right) + \alpha I_1\left(\frac{\alpha}{2}\right) \right]^2} \tag{12}$$

* Received by the PGIT, December 19, 1960; revised manuscript received, March 14, 1961.

[1] S. O. Rice, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332, July, 1944; vol. 24, pp. 46–156; January, 1945.

[2] D. Middleton, "An Introduction to Statistical Communication Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1960.

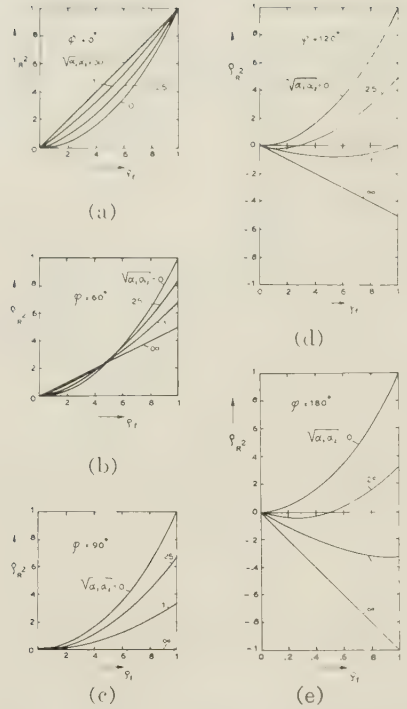

Fig. 1—Correlation coefficient ρ_{R^2} of the output from a square-law detector plotted vs the correlation coefficient ρ_f of the undetected signals. The parameters are: the phase angle φ and the geometric mean of the SNR's α_1 and α_2.

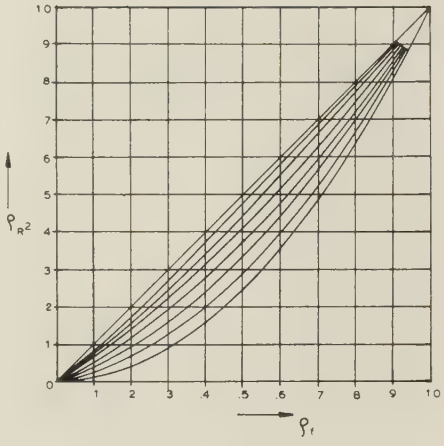

Fig. 2—Correlation coefficient ρ_{R^2} of the output from a square-law detector plotted vs the correlation coefficient ρ_f of the undetected signals, for the special case $\varphi = 0$. The parameter $\sqrt{\alpha_1\alpha_2}$ has the values, starting from the lower curve: 0, 0.1, 0.25, 0.5, 1, 2, 4, and ∞.

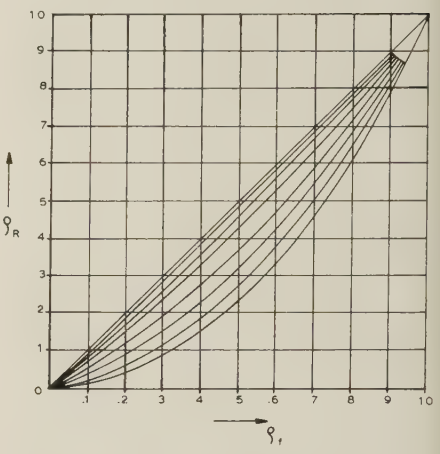

Fig. 3—The correlation coefficient ρ_R of the output from a linear detector plotted vs the correlation coefficient ρ_f of the undetected signals, for the special case $\varphi = 0$ and $\alpha_1 = \alpha_2 = \alpha$. The parameter α has the values, starting from the lower curve: 0, 0.1, 0.25, 0.5, 1, 2, 4, and ∞.

which is valid for small values of ρ_f, $\rho_f \ll 1$.

Here, $E(\rho_f)$ and $K(\rho_f)$ are the elliptic integrals, and $I_0(\alpha/2)$ and $I_1(\alpha/2)$ are the modified Bessel functions. Eq. (12) is valid with good accuracy even for relatively large values of ρ_f. For any value of the parameters, the error is less than 9 per cent.

Comparison of (8), (10) and (11) shows that for zero SNR, ρ_R is less than ρ_{R^2} by a factor which turns out to be between 0.91 and 1, and for large SNR, ρ_R is larger than ρ_{R^2} by a factor approaching unity.

Incoherent Signals

Up to this point, we have assumed that the two signals have the same frequency; *i.e.*, the two signals are coherent. Two signals with different frequencies may be expressed mathematically as having equal frequencies and a phase difference varying linearly with time. Thus the angle $\phi_1 - \phi_2$, and hence φ is uniformly distributed in the interval from zero to 2π. The results derived for coherent signals may therefore be applied to incoherent signals by averaging with respect to φ.

Carrying out the averaging process on the results for coherent signals, we obtain:

Square-law detector:

$$\rho_{R^2} = \frac{\rho_f^2}{1 + 2\sqrt{\alpha_1\alpha_2}}. \qquad (13)$$

Linear detector, small values of α:

$$\rho_R \approx \frac{2}{(4-\pi)(1+\alpha)}\left\{2E(\rho_f) - (1-\rho_f^2)K(\rho_f) - \frac{\pi}{2} + \alpha\left[E(\rho_f) - \frac{\pi}{2}\right]\right\}. \qquad (14)$$

Linear detector, large values of α:

$$\rho_R \approx \frac{\rho_f^2}{8\alpha - 2}. \qquad (15)$$

Linear detector, small values of ρ_f:

$$\rho_R \approx \frac{\frac{\rho_f^2}{4} I_0\left(\frac{\alpha}{2}\right)^2}{4e^{\alpha}(1+\alpha)/\pi - \left[(1+\alpha)I_0\left(\frac{\alpha}{2}\right) + \alpha I_1\left(\frac{\alpha}{2}\right)\right]^2}. \qquad (16)$$

From (13) and (15), we observe that for equal and very large values of the SNR's ρ_{R^2} exceeds ρ_R by a factor 4. For this type of signal, the linear and the square-law detector behave quite differently.

Kjell Blötekjaer
Div. for Radar
Norwegian Defense Res. Establishment
Bergen, Norway

Golay's Complementary Series*

Golay, in a recent paper,[1] introduced the notion of *complementary sequences* of 0's and 1's. He states as a result obtained by trial that complementary sequences of length 18 do not exist. We prove this result.

In order to define complementary sequences, let $S = (s_1, \cdots, s_n)$ be any sequence of 0's and 1's. We first define

$L_i(S)$ = the number of values of j for which $s_j = s_{j+i}$,
= the number of like pairs of s_j's which are separated by distance i,

$U_i(S)$ = the number of values of j for which $s_j \neq s_{j+i}$,
= the number of unlike pairs of s_j's which are separated by distance i.

We note that

$$L_i(S) + U_i(S) = \begin{cases} n - i & \text{for } 1 \leqq i \leqq n-1, \\ 0 & \text{for } n \leqq i. \end{cases}$$

Now let $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_n)$. We define *equivalence*:

$$A \equiv B$$

if

$$L_i(A) = L_i(B)$$

and

$$U_i(A) = U_i(B)$$

for all i. We define *complementary sequences*:

$$A \sim B$$

if

$$L_i(A) = U_i(B)$$

and

$$U_i(A) = L_i(B)$$

for all i. We see from the equation above that $L_i(A) = U_i(B)$ for all i implies $A \sim B$. We also see that $\equiv$ is an equivalence relation, that $\sim$ is symmetric, and that equivalent sequences behave identically with respect to $\sim$.

Golay proves many facts about complementary series. In particular, he investigates the values of n for which they can exist. His theorems settle the existence question for $n < 18$ and for many larger values as well. He asserts that "it has been verified by trial that complementary series do not exist for $n = 18$." The purpose of this note is to give a reasonably compact proof of this fact.

* Received by the PGIT, January 5, 1961.

[1] M. J. E. Golay, "Complementary series," IRE Trans. on Information Theory, vol. IT-7, pp. 82–87; April, 1961.

We let

$\bar{L}_i(S) = L_i(S) + L_{2i}(S) + L_{3i}(S) + \cdots$
= the number of like pairs whose subscripts are congruent mod i,

$\bar{U}_i(S) = U_i(S) + U_{2i}(S) + U_{3i}(S) + \cdots$
= the number of unlike pairs whose subscripts are congruent mod i.

For $1 \leqq j \leqq i$, we let

$p_{ij}(S) = s_j + s_{j+i} + s_{j+2i} + \cdots$
= the number of values of k such that $k \equiv j \pmod{i}$ and $s_k = 1$.

Thus p_{11} = the number of 1's in S, p_{21} = the number of 1's in odd positions, p_{22} = the number of 1's in even positions, etc. Suppose that $i \mid n$, and that $k = n/i$. Then it is easy to see that

$$\bar{L}_i(S) = \sum_{j=1}^{i} \left[\binom{p_{ij}}{2} + \binom{k - p_{ij}}{2} \right],$$

$$\bar{U}_i(S) = \sum_{j=1}^{i} p_{ij}(k - p_{ij}).$$

As special cases, we see that

$\bar{L}_1(S)$ = the number of like pairs in S

$$= \binom{p_{11}}{2} + \binom{n - p_{11}}{2},$$

$\bar{U}_1(S)$ = the number of unlike pairs in S

$$= p_{11}(n - p_{11}).$$

For brevity, let

$$f(p, k) = \binom{p}{2} + \binom{k - p}{2},$$

$$g(p, k) = p(k - p).$$

Now suppose that A and B are sequences of length 18 and that $A \sim B$. We wish to reach a contradiction. We first let

$$\bar{L}_i = \bar{L}_i(A), \qquad \bar{U}_i = \bar{U}_i(B),$$

$$p_{ij} = p_{ij}(A), \qquad q_{ij} = p_{ij}(B).$$

Then we have (if $i \mid 18$)

$$\bar{L}_i = \sum_j f(p_{ij}, 18/i) = \bar{U}_i = \sum_j g(q_{ij}, 18/i).$$

This equation is our main tool. We shall successively let $i = 1, 2, 3, 6$. In the case $i = 1$, we find that

$$\binom{p_{11}}{2} + \binom{18 - p_{11}}{2} = q_{11}(18 - q_{11}).$$

Elementary algebra leads to

$$18 = (18 - p_{11} - q_{11})^2 + (p_{11} - q_{11})^2.$$

[This is Golay's (7).] Thus

$$18 - p_{11} - q_{11} = \pm 3,$$

$$p_{11} - q_{11} = \pm 3.$$

Then $\{p_{11}, q_{11}\} = \{6, 9\}$ or $\{12, 9\}$, where the curly brackets indicate unordered sequences. We can arrange that $q_{11} = 9$ by interchanging the sequences used for A and B if necessary. Similarly, we can arrange that $p_{11} = 6$ by "altering" the sequence used for A if necessary. We define alteration thus:

$$0' = 1, \qquad 1' = 0,$$

$$(s_1, \cdots, s_n)' = (s_1', \cdots, s_n'),$$

the alteration of S is S'. Clearly, $S \equiv S'$ and $p_{11}(S') = n - p_{11}(S)$. Thus if $p_{11} = 12$, then by replacing the sequence used for A by its alteration, we still have a complementary pair of sequences and $p_{11} = 6$.

We have seen that without real loss of generality, we may assume that $p_{11} = 6$ and $q_{11} = 9$. Next we wish to let $i = 2$. Obviously, we have

$$p_{21} + p_{22} = p_{11} = 6,$$

$$q_{21} + q_{22} = q_{11} = 9.$$

We consider all possible cases in the following small tables:

p	$f(p, 9)$
0	36
1	28
2	22
3	18
4	16
5	16
6	18

$\{p_{21}, p_{22}\}$	$f(p_{21}, 9) + f(p_{22}, 9) = \bar{L}_2$
6, 0	52
5, 1	44
4, 2	38
3, 3	36 ✓

q	$g(q, 9)$	$\{q_{21}, q_{22}\}$	$g(q_{21}, 9) + g(q_{22}, 9) = \bar{U}_2$
0	0	9, 0	0
1	8	8, 1	16
2	14	7, 2	28
3	18	6, 3	36 √
4	20	5, 4	40
5	20		
6	18		
7	14		
8	8		
9	0		

Thus, $\bar{L}_2 = \bar{U}_2$ implies that $\{p_{21}, p_{22}\} = \{3, 3\}$ and $\{q_{21}, q_{22}\} = \{6, 3\}$. We can arrange that $q_{21} = 6$ and $q_{22} = 3$ by writing the sequence used for B backwards if necessary. It is clear that a sequence written backwards is equivalent to the same sequence written forward, and that this rearrangement of B interchanges the values of q_{21} and q_{22}.

We now know that without real loss of generality, we may assume

$$p_{11} = 6; \quad p_{21} = 3, \quad p_{22} = 3,$$
$$q_{11} = 9; \quad q_{21} = 6, \quad q_{22} = 3.$$

Next we take $i = 3$. Obviously

$$p_{31} + p_{32} + p_{33} = p_{11} = 6,$$
$$q_{31} + q_{32} + q_{33} = q_{11} = 9.$$

Furthermore, p_{3j} and q_{3j} are $\leqq 18/3 = 6$. The following tables include all possibilities:

p	$f(p, 6)$	$\{p_{31}, p_{32}, p_{33}\}$	$\bar{L}_3$
0	15	6, 0, 0	45
1	10	5, 1, 0	35
2	7	4, 2, 0	29
3	6	4, 1, 1	27 *II*
4	7	3, 3, 0	27 *II*
5	10	3, 2, 1	23
6	15	2, 2, 2	21 *I*

q	$g(q, 6)$	$\{q_{31}, q_{32}, q_{33}\}$	$\bar{U}_3$
0	0	6, 3, 0	9
1	5	6, 2, 1	13
2	8	5, 4, 0	13
3	9	5, 3, 1	19
4	8	5, 2, 2	21 *I*
5	5	4, 4, 1	21 *I*
6	0	4, 3, 2	25
		3, 3, 3	27 *II*

Using $\bar{L}_3 = \bar{U}_3$, we find that the following two cases are possible:

	$\{p_{31}, p_{31}, p_{33}\}$	$\{q_{31}, q_{32}, q_{33}\}$
I	2, 2, 2	*i* 5, 2, 2
		ii 4, 4, 1
II	*i* 4, 1, 1	3, 3, 3
	ii 3, 3, 0	

Similarly, in the case $i = 6$, we have

$$\sum_j p_{6j} = p_{11} = 6, \quad \sum_j q_{6j} = q_{11} = 9,$$

and p_{6j} and q_{6j} are $\leqq 18/6 = 3$. The following tables include all cases.

p	$f(p, 3)$	$\{p_{61}, \cdots, p_{66}\}$	$\bar{L}_6$
0	3	330000	18
1	1	321000	14
2	1	311100	12 *D*
3	3	222000	12 *D*
		221100	10 *C*
		211110	8 *B*
		111111	6 *A*

q	$g(q, 3)$	$\{q_{61}, \cdots, q_{66}\}$	$\bar{U}_6$
0	0	333000	0
1	2	332100	4
2	2	331110	6 *A*
3	0	322200	6 *A*
		322110	8 *B*
		321111	10 *C*
		222210	10 *C*
		222111	12 *D*

Using $\bar{L}_6 = \bar{U}_6$, we see that there are four cases:

	$\{p_{61}, \cdots, p_{66}\}$	$\{q_{61}, \cdots, q_{66}\}$
A	111111	*a* 331110
		b 322200
B	211110	322110
C	221100	*a* 321111
		b 222210
D	*a* 311100	222111
	b 222000	

Not let us arrange the p_{6j} and the q_{6j} into matrices:

$$\begin{bmatrix} p_{61} & p_{65} & p_{63} \\ p_{64} & p_{62} & p_{66} \end{bmatrix}, \qquad \begin{bmatrix} q_{61} & q_{65} & q_{63} \\ q_{64} & q_{62} & q_{66} \end{bmatrix}.$$

It is obvious that the row sums must be $p_{21} = 3$, $p_{22} = 3$ and $q_{21} = 6$, $q_{22} = 3$, respectively. It is also obvious that the column sums, in some order, must be p_{31}, p_{32}, p_{33} and q_{31}, q_{32}, q_{33}, respectively. Thus combining the cases for the p_{3j}, q_{3j} and the p_{6j}, q_{6j}, we get the following tables of all possible cases. We use $\begin{pmatrix} a \\ b \end{pmatrix}$ to denote a column vector. We write "impossible (col)" if it is impossible to arrange the 6 numbers (either the p_{6j} or the q_{6j}) so as to have the correct column sums. We write "impossible (row)" if it *is* possible to make the column sums correct, but not simultaneously possible to make the row sums correct.

	$\left\{\begin{pmatrix} p_{61} \\ p_{64} \end{pmatrix}, \quad \begin{pmatrix} p_{65} \\ p_{62} \end{pmatrix}, \quad \begin{pmatrix} p_{63} \\ p_{66} \end{pmatrix}\right\}$
IA	$\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}$
IB	impossible (row)
IC	$\begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\2\end{pmatrix}$
IDa	impossible (col)
IDb	impossible (row)
IIiA	impossible (col)
IIIiA	impossible (col)
IIiB	impossible (col)
IIIiB	impossible (col)
IIiC	$\begin{pmatrix}2\\2\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}$
IIic	$\begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}1\\2\end{pmatrix}, \quad \begin{pmatrix}0\\0\end{pmatrix}$
IIiDa	$\left\{\begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}\right.$ $\left.\begin{pmatrix}1\\3\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}\right.$
IIiDb	impossible (col)
IIIiDa	impossible (col)
IIIiDb	impossible (col)

	$\left\{\begin{pmatrix} q_{61} \\ q_{64} \end{pmatrix}, \quad \begin{pmatrix} q_{65} \\ q_{62} \end{pmatrix}, \quad \begin{pmatrix} q_{63} \\ q_{66} \end{pmatrix}\right\}$
IiAa	impossible (col)
IiAb	$\begin{pmatrix}2\\3\end{pmatrix}, \quad \begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}2\\0\end{pmatrix}$
IiiAa	$\begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}$
IiiAb	impossible (col)
IiB	$\begin{pmatrix}3\\2\end{pmatrix}, \quad \begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}$
IiiB	$\begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}2\\2\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}$
IiCa	impossible (row)
IiCb	impossible (col)
IiiCa	impossible (col)
IiiCb	impossible (row)
IiD	impossible (col)
IiiD	impossible (col)
IIAa	impossible (col)
IIAb	impossible (col)
IIB	$\begin{pmatrix}3\\0\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}1\\2\end{pmatrix}$
IICa	impossible (col)
IICb	impossible (col)
IID	$\begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}$

We now eliminate all cases which are impossible on either side and combine the tables into the following single table.

	$\left\{\begin{pmatrix} p_{61} \\ p_{64} \end{pmatrix}, \quad \begin{pmatrix} p_{65} \\ p_{62} \end{pmatrix}, \quad \begin{pmatrix} p_{63} \\ p_{66} \end{pmatrix}\right\}$	$\left\{\begin{pmatrix} q_{61} \\ q_{64} \end{pmatrix}, \quad \begin{pmatrix} q_{65} \\ q_{62} \end{pmatrix}, \quad \begin{pmatrix} q_{63} \\ q_{66} \end{pmatrix}\right\}$
IA	$\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}$	ib $\begin{pmatrix}2\\3\end{pmatrix}, \quad \begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}2\\0\end{pmatrix}$
		iia $\begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}$
IID	ia1 $\begin{pmatrix}3\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}$
	ia2 $\begin{pmatrix}1\\3\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}1\\0\end{pmatrix}$	

Now we are ready for our final step. Golay proves [see his (4)] that if $A \sim B$, then

$$a_i + a_{n+1-i} + b_i + b_{n+1-i} = \text{odd number} = 1 \quad \text{or} \quad 3$$

In our case, if we define $h(i)$ in the following equation, we have

$$h(i) = a_i + a_{19-i} + b_i + b_{19-i} = \text{odd number}.$$

Therefore

$$p_{61} + p_{66} + q_{61} + q_{66} = h(1) + h(7) + h(13) = \text{odd number},$$

$$p_{62} + p_{65} + q_{62} + q_{65} = h(2) + h(8) + h(14) = \text{odd number},$$

$$p_{63} + p_{64} + q_{63} + q_{64} = h(3) + h(9) + h(15) = \text{odd number}.$$

In case IA in the last table, we must have

$$p_{61} + p_{66} = 2, \qquad p_{62} + p_{65} = 2, \qquad p_{63} + p_{64} = 2$$

all even. Hence, the corresponding sums of the q_{6j} must all be odd. But in the two possibilites for the q's in case IA, either. five of the q_{6j} are even, or five are odd. Hence, only *one* of the sums

$$q_{61} + q_{66}, \qquad q_{62} + q_{65}, \qquad q_{63} + q_{64}$$

can be odd. Thus, case IA is impossible. In case IID in the last table, we must have

$$q_{61} + q_{66} = 3, \qquad q_{62} + q_{65} = 3, \qquad q_{63} + q_{64} = 3,$$

all odd. Hence, the corresponding sums of the p_{6j} must all be even. Now each of the sums

$$p_{61} + p_{66}, \qquad p_{62} + p_{65}, \qquad p_{63} + p_{64},$$

involves one term from the top of a vector and one from the bottom. Consequently, it is easy to see (for both possibilities in case IID) that these sums in some order are {4, 1, 1}, and hence are not all even. This completes the proof that it is impossible to have complementary series of length 18.

Joseph B. Kruskal
Bell Telephone Labs., Inc.
Murray Hill, N. J.

On "Upper Bounds for Error Detecting and Correcting Codes of Finite Length"*

In a recent article,[1] Wax cited the results of Laemmel that the best value for the number of sequences of length 14 which correct two errors is "48(?)." A better value is 64. To see this, consider a code[2] developed by Bose and Chaudhuri of length 15 with 7 information bits which is able to correct 2 errors. Leaving off one information bit, this would be a code of length 14 with 6 information bits, or 64 code points.

There are several entries in Table I of Wax's paper which are blank. A MacDonald[3] maximum-minimum distance code will give 8 code points with length 17 capable of correcting four errors. There is also an optimum group code with $2^9 = 512$ code points of length 17 that is capable of correcting 2 errors.[4]

These results are consistent with the bounds Wax found. It should be noted that these results appeared after Wax's paper was published.

F. F. Sellers
Reliability Technology
IBM Product Dev. Lab.
Poughkeepsie, N. Y.

* Received by the PGIT, January 6, 1961.

[1] N. Wax, "On upper bounds for error detecting and correcting codes of finite length," IRE Transactions on Information Theory, vol. IT-5, pp. 168–174; December, 1959.

[2] R. C. Bose and D. K. Ray-Chaudhuri, "On a class of error correcting binary group codes," *Information and Control*, vol. 3, pp. 68–79; March, 1960.

[3] J. E. MacDonald, "Design methods for maximum minimum distance error detecting and correcting codes," *IBM J. Res. & Dev.*, vol. 4, pp. 43–57; January, 1960.

[4] W. W. Peterson, "Error Detecting and Correcting Codes," Technology Press and John Wiley and Sons, Inc., New York, N. Y.; 1961. A table of optimum group codes is given in Chapter 5 of the book.

Corrections

Phillip Bello, author of "On the Approach of a Filtered Pulse Train to a Stationary Gaussian Process," which appeared on pp. 144–149 of the July, 1961, issue of these Transactions, has called the following to the attention of the *Editor*.

On p. 149, first column, (58) should read:

$$\sum_{m=-\infty}^{\infty} |\bar{C}_m^* \cdot \bar{\lambda}|^2 = \sum_{p,q=1}^{N} \lambda_p \lambda_q^* \sum_{m=-\infty}^{\infty} C_{mp}^* C_{mq} = \sum_{p,q=1}^{N} R_z(t_p - t_q)\lambda_p \lambda_q^*$$

$$\sum_{m=\infty}^{\infty} (\bar{C}_m^* \cdot \bar{\lambda})^2 = \sum_{p,q=1}^{N} \lambda_p \lambda_q \sum_{m=-\infty}^{\infty} C_{mp} C_{mq} = \sum_{p,q=1}^{N} \tilde{R}_z(t_p - t_q)\lambda_p \lambda_q = 0 \tag{58}$$

On the same page, in the line following (56), I_1 should be replaced by script $\mathfrak{I}_1$. Similarly, the I_1 in (59) should be script $\mathfrak{I}_1$.

Carl W. Helstrom, author of "Maximum-Weight Group Codes for the Balanced M-ary Channel," which appeared on pp. 550–555 of the December, 1960, issue of these Transactions, has called the following to the attention of the *Editor*.

After the author returned the proofs of the above paper, Tables II, III, and IV were altered without his knowledge into a form that may be misunderstood. In Table II, for $k = 4$, $8 \leq h \leq 14$, "1, 2, 3, 4, 124, 134, 234" should have been a ditto mark. The same holds in Tables III and IV for "01" for $k = 2$, $h > 1$, and for "001" for $k = 3$, $h > 1$.

To find the proper columns of the MRT for any value of h, the column whose label appears after the + sign should be appended to the set of columns used for $h - 1$. For example, the labels of the columns of the MRT to be used for a code with $M = 4$, $k = 3$, $h = 5$ are 001, 010, 100, 111, $1\beta\alpha$.

The weights w_i given in Table III for $M = 3$, $k = 3$, $h = 2$ should read "$0^2 1^{12} 2^{12}$."

D. C. Youla, author of "On the Factorization of Rational Matrices," which appeared on pp. 172–189 of the July, 1961, issue of these Transactions, has called the following to the attention of the *Editor*.

On p. 172, second column, line 24 from the top:

"$b_{rk}(p)$" should read "$f_{rk}(p)$."

On p. 180, first column, second line above (89):

"$2\delta_2 = 2$." should read "$2\delta_1 = 2\delta_2 = 2\delta_3 = 2$."

On p. 188, first column, in (179):

"$\Psi'_*(p)$" should read "$\Psi_*(p)$."

On p. 188, first column, in the second line above (180):

"$\hat{V}(p)$" should read "$\hat{G}(p)$."

On p. 189, first column, in line 12 from the top:

"(178)" should read "(179)."

Contributors

Phillip Bello (S '52—A '55) was born in Lynn, Mass., on October 22, 1929. He received the B.S.E.E. degree from Northeastern University, Boston, Mass., in 1953, and the M.S. and D.Sc. degrees in 1955 and 1959, respectively, from the Massachusetts Institute of Technology, Cambridge, Mass.

From 1955 to 1957, he was a Research Associate and then Assistant Professor of communications at Northeastern University, teaching courses in linear system analysis and statistical communication theory. Also during this time he was involved in classified research in statistical communication theory. From 1956 to 1958, he was employed by Dunn Engineering Associates, Cambridge, Mass., where he was engaged in analytical studies associated with various aspects of statistical radar theory.

From 1958 to September, 1961 he was employed at the Applied Research Laboratory of Sylvania Electronic Systems, Waltham, Mass., working on various aspects of statistical communication theory. Since October 1961, he has been employed by Adcom, Inc., Cambridge, Mass.

Dr. Bello is a member of Tau Beta Pi, Sigma Xi, and Eta Kappa Mu.

❖

Robert C. Heimiller (S '54—A '57—M '60) was born in Buffalo, N. Y., on August 6, 1932. He received the B.E.E. degree from the University of Detroit, Mich., in 1955, and the M.S.E.E. degree from the University of Southern California, Los Angeles, in 1957. Since 1957, he has been engaged in graduate studies at the University of California at Los Angeles.

From 1953 to 1955, he worked at Bell Aircraft Corporation, Buffalo, as an Engineer Trainee under the Cooperative Engineering Plan. In 1955, he joined Hughes Aircraft Company, Culver City, Calif., where for two years he was assigned to the Rotational Training Plan. In this capacity, he engaged in analog simulation of flight control systems; laboratory tests of range tracking circuits; support for the flight test program of a prototype fire control system; and design of waveguide transitions and ferrite switches. In 1957, he became Staff Engineer, Analysis Staff, Signal Processing and Display Laboratory, Aerospace Group, engaged in radar system analysis and design. His work has included: noise and clutter analysis; detection and tracking range predictions; advanced signal processing techniques; application of information, detection and matched filter theories; and ground, sea clutter, and propagation phenomena. He is presently engaged in study and development of concepts for advanced radar techniques, and in radar system design and evaluation.

Mr. Heimiller is a member of Tau Beta Pi and Eta Kappa Nu.

Tadao Kasami was born in Kobe, Japan, on April 12, 1930. He received the B.E. and the M.E. degrees in electrical engineering in 1958 and 1960, respectively, from the University of Osaka, Japan, and is now in the graduate division working towards the Ph.D. degree in electrical engineering.

Mr. Kasami is a member of the Institute of Electrical Communication Engineers and the Information Processing Society, both of Japan.

❖

Ludwik Kurz (M '60) was born in Lwow, Poland, on January 24, 1926. He received the B.E.E. and M.E.E. degrees from the College of the City of New York in 1951 and 1955, respectively, and the Eng. Sc. D. degree from New York University, New York, N. Y., in 1961. He was awarded the National Science Foundation Science Faculty Fellowship for the period from September, 1959, to January, 1961.

From 1951 to 1959, he taught graduate and undergarduate courses in electrical engineering at CCNY. Since 1959, he has been doing research in the field of optimum detection of signals in noise and allied topics. At present, he holds the position of Research Scientist at New York University, College of Engineering; he also teaches graduate courses in electrical engineering.

Dr. Kurz is a member of Eta Kappa Nu.

❖

J. E. Meggitt was born in London, England, on August 31, 1931. He received the M.A. and Ph.D. degrees from Emmanuel College, Cambridge University, Cambridge, England, in 1950. He began his career in mathematics there, applying himself to some academic problems in the quantum theory of fields. He won a Smith Prize in 1955 and became a Research Fellow of the College.

He then entered the aircraft industry, becoming interested in computers and, in particular, in problems associated with the numerical control of machine tools. In 1959, he joined the IBM British Laboratories, Winchester, Hants, England, where he is currently engaged in the design of computer systems.

❖

Nils J. Nilsson (S '54—M '58) was born in Saginaw, Mich., on February 6, 1933. He received the M.S. and Ph.D. degrees in electrical engineering in 1956 and 1958, respectively, from Stanford University, Stanford, Calif., where, as a graduate student, he received a National Science Foundation Fellowship. His graduate field of study was in the application of statistical techniques to radar and communications problems.

In July, 1961, he completed a three-year term of active duty as a lieutenant in the USAF, stationed at Rome Air Development Center, Griffiss Air Force Base, Rome, N. Y. His duties entailed research in advanced radar techniques, signal analysis and the application of statistical techniques to radar problems. He has written several papers on various aspects of radar signal processing. While stationed at RADC, he held an appointment as Lecturer in the Electrical Engineering Department of Syracuse, University, Syracuse, N. Y

Since August, 1961, he has been on the staff of the Stanford Research Institute, Menlo Park, Calif.

Dr. Nilsson is a member of Sigma Xi and Tau Beta Pi.

❖

Frank Tung (S '55—M '57) was born in Chekiang, China, on August 4, 1933. He received the B.S.E.E. and M.S.E.E. degrees from the Massachusetts Institute of Technology, Cambridge, in 1955 and 1956, respectively, and the Eng. Sc. D. degree from Columbia University, New York, N. Y., in 1961.

He was a Research Assistant at the Research Laboratory of Electronics, MIT, from 1955 to 1956 and in the Department of Electrical Engineering of Columbia University from 1958 to 1961. From 1956 to 1958, he was with the Product Development Laboratory of IBM, Poughkeepsie, N. Y., working primarily on the system planning of digital computers. He joined the IBM Research Laboratory, San Jose, Calif. in June, 1961 where he is presently doing research on signal theory and adaptive systems.

Dr. Tung is a member of Sigma Xi and Eta Kappa Nu.

❖

Ralph J. Schwarz (S '43—A '44—M '49—SM '54) was born in Hamburg, Germany, on June 13, 1922. He received the B.S., M.S., and Ph.D. degrees in electrical engineering from Columbia University, New York, N. Y., in 1943, 1944, and 1949, respectively. He also did additional graduate work at the Polytechnic Institute of Brooklyn, N. Y., and New York University.

He has taught at Columbia University and (from 1955 to 1956) at the University of California at Los Angeles. He has also been active as a consultant in noise theory and system analysis, including one year at the Ramo-Wooldridge Corporation. His research interests include nonlinear circuits and random processes. Currently, he is Professor and Chairman of the Department of Electrical Engineering at Columbia University.

Dr. Schwarz is a member of the AIEE, American Physical Society, Tensor Club of Great Britain, Tau Beta Pi, Sigma Xi, Eta Kappa Nu, and Pi Mu Epsilon.

Abstracts

This Section of the issue is devoted to abstracts of material which may be of interest to PGIT members. Sources are Government, Industrial and University reports, and books and journals published outside of the United States. Readers familiar with material of this nature which is suitable for abstracting are requested to communicate the pertinent information to one of the Editors or Correspondents listed below.

An Analog Pulse Correlator—T. Bartokowski and J. Seidler (in Polish). (*Zeszyty Naukowe Politechniki Gdańskiej*, no. 20, pp. 49–65; 1960.)

A description is given of an analog pulse correlator for the band extending from 50 cps to 20 kc. The method applied is based on the sampling principle. Two time-shifted samples modulate the height and width of pulses, and a voltage proportional to the correlation function is obtained by integration of the train of such doubly modulated pulses. The equipment is suitable for measuring both auto- and cross-correlation functions.

Frequency-Time Transposition for the Measurement of an Unknown Frequency, II—R. H. Baumann (in French). (*Ann. de Radioélec.*, vol. 16, pp. 69–92; January, 1961.)

In Part I of this investigation (see July, 1961, Abstracts), it was shown that the circulating sinusoidal signals in a closed-loop system, which consists of a delay line and a modulator, are transformed into impulses whose time shift is a measure of the unknown Doppler frequency to be determined.

In Part II, the system limitations are considered; more specifically, an imperfect system with spurious signals in the delay line and in the modulator is treated theoretically. This analysis provides a good explanation of the anamolous experimental results described in Part I.

In conclusion, several methods are proposed for improving the system stability and for minimizing the undesirable effect of spurious signals. The maximum number of recirculations obtained experimentally with one of these methods was 400, which corresponds to a twenty-fold improvement in the signal-to-noise amplitude ratio, and to the determination of the Doppler frequency to within 0.5 per cent.

Sequential Signal Detector, Rayleigh Case—J. W. Caspers (in English). (Navy Electronics Lab., San Diego, Calif., Rept. No. 730; November 7, 1956.)

A detector based on the sequential probability ratio test has been examined theoretically. It is shown that it will respond to signals more rapidly, on the average, than the best nonsequential detector. Equations and curves are given which predict the performance of the sequential detector and compare it with that of the Neyman-Pearson detector. Further analyses and experimentation are outlined which are directed toward a more comprehensive understanding of the sequential detector, toward its simplification, and toward an estimation of its value as a SNR estimator.

An Experimental Sequential Detector—G. M. Dillard and R. E. Simmons (in English). (Navy Electronics Lab., San Diego, Calif., Rept. No. 999; November 8, 1960.)

A binomial sequential detector has been developed to facilitate laboratory experiments in the development of radar detection techniques based on statistical methods. The system has been found to be capable of performance comparable with predictions made by exact equations and certain approximations valid for large sample cases. A device for generating a random sequence of ones and zeros, with the probability of a one fixed, has also been built. Extensive laboratory experiments employing both equipments are described.

Television Band Compression by Contour Interpolation—D. Gabor and P. C. J. Hill (in English). (*Proc. IEE.*, vol. 108, pt. B, pp. 303–315; May, 1961.)

It is proposed to economize bandwidth by transmitting a proportion only of the scanning lines in the picture: in the simplest case alternate lines would be transmitted, but in more extreme applications only one in four or one in eight. The missing lines are to be reconstructed in the receiver by a process of interpolation which is sensitive to contours (edges) and inserts the transition in the interpolated scanning line in such a way as to maintain straightness of contour. Contours are identified by means of a differentiating circuit, followed by further gating and clipping circuits to produce a sharp pulse at the leading edge of each contour. If a, b, and c are successive lines, then a and c must be scanned simultaneously so that b can be interpolated, and this requires that a be stored until the whole of c has been scanned. When the scanning spot on one line reaches a contour it stops, and the scanning spot on the other line is caused to move at double velocity until it also reaches the contour. The scanning velocity for the interpolated line is the mean of the two. The brightness of the interpolated line is fixed by weighting the brightness of each of the adjacent lines with the velocity of its scanning spot and taking the mean. Thus, when both spots are moving, the brightness of the interpolated line is the mean of those of adjacent lines, but when one spot stops the brightness of the interpolated line is that of the spot which is still moving, *i.e.*, of the picture before the edge. The system was tried out on a slow-speed model similar to a facsimile system, with transmission of a photograph and reception on photographic paper. The velocity modulation of the spot was carried out mechanically and controlled by a pilot spot on each of the pair of lines scanning ahead of the transmitting spots.

A scheme is proposed for an electronic realization of such a

system. At the transmitter, the picture is to be produced on a double-beam cathode-ray tube and transferred to a camera tube. The picture may be read out two lines at a time from a single-beam camera tube by oscillating the beam between two lines; and the change of sweep velocity required on encountering a contour can be superimposed on the movement of the beam. The receiver for a 2:1 compression system would require two storage tubes; for 8:1 compression the receiver would require six camera tubes and six cathode-ray tubes of which one would be double-beam and two would have variable-velocity scanning. Line-by-line rate equalization methods should give a compression of 3:1 and by combining this with a compression of either 4:1 or 8:1 by means of interpolation, the over-all compression should be either 12:1 or 24:1.

Waiting System With Full Availability Where the Holding Time of the Incoming Trunk is Larger Than That of the Outgoing Trunk—E. Gambe (in Japanese). (*J. Inst. Elec. Commun. Engrs. (Japan)*, vol. 44, pp. 227–233; February, 1961.)

Probabilities and other related formulas of various simultaneous connections of a waiting system with full availability, where the holding time of incoming trunk is longer than that of outgoing trunk, are derived for the stationary state under the assumption that the occurrence of calls is Poisson distributed and the holding time is exponentially distributed. A method of calculation of these for the multistage waiting system by applying known formulas is discussed, and it is shown that the numerical tables and curves for an ordinary waiting system with full availability are applicable to the multistage waiting system under a certain constraint on the calculation of calls.

On the General Definition of the Amount of Information—I. M. Gel'fand, A. N. Kolmogorov, and A. M. Yaglom (in Russian). (*Doklady Akad. Nauk S.S.S.R.*, vol. 111, no. 4, pp. 745–748; 1956.)

The system S of "random events" $A, B, C, \ldots$ is supposed to be a Boolean algebra over which a normed, non-negative function $P(A)$ is defined. $P(A)$ is additive for nonoverlapping A, B. An "experiment" is identified with a subalgebra of the algebra S. Otherwise speaking, a subalgebra A consists of all events whose results become known after the end of the experiment. If subalgebras A and L are finite, then the amount of information contained in the results of experiment A about the results of experiment L is given by Shannon's formula

$$I(\mathsf{A}, \mathsf{L}) = \sum_{i,j} P(A_i, B_j) \log \frac{P(A_iB_j)}{P(A_i)P(B_j)}.$$

In the general case, it is natural to put $I(\mathsf{A}, \mathsf{L}) = \sup^{\mathsf{A}_1 \subseteq \mathsf{A}, \mathsf{L}_1 \subseteq \mathsf{L}} I(\mathsf{A}_1, \mathsf{L}_1)$, where A_1, respectively L_1, are finite subalgebras of A, respectively L. Thus, the defined amount of information has the following known properties. 1) $I(\mathsf{A}, \mathsf{L}) = I(\mathsf{L}, \mathsf{A})$. 2) $I(\mathsf{A}, \mathsf{L}) = 0$ if, and only if, A and L are independent. 3) If the smallest subalgebras, $[\mathsf{A}_1 \cup \mathsf{L}_1]$, $[\mathsf{A}_2 \cup \mathsf{L}_2]$, which contain $\mathsf{A}_1 \cup \mathsf{L}_1$ and $\mathsf{A}_2 \cup \mathsf{L}_2$ are independent, then $I([\mathsf{A}_1 \cup \mathsf{A}_2], [\mathsf{L}_1 \cup \mathsf{L}_2]) = I(\mathsf{A}_1, \mathsf{L}_1) + I(\mathsf{A}_2, \mathsf{L}_2)$. 4) If $\mathsf{A}_1 \subseteq \mathsf{A}_2$, then $I(\mathsf{A}_1, \mathsf{L}) = I(\mathsf{A}_2, \mathsf{L})$.

The following theorems are considered to be almost obvious. *Theorem 1*: if the algebra $\mathsf{A}_1 \subseteq \mathsf{A}$ is everywhere dense over A in the sense of the metric $\rho(A, B) = P(AB' \cup A'B)$, then $I(\mathsf{A}_1, \mathsf{L}) = I(\mathsf{A}, \mathsf{L})$. *Theorem 2*: If $\mathsf{A}_1 \subseteq \mathsf{A}_2 \subseteq \cdots \mathsf{A}_n \subseteq \cdots$ and $\mathsf{A} \subseteq \cup \mathsf{A}_n$, then $I(\mathsf{A}, \mathsf{L}) = \lim_{n\to\infty} I(\mathsf{A}_n, \mathsf{L})$. *Theorem 3*: if the sequence of distributions P^n converges on $[\mathsf{A} \cup \mathsf{L}]$ towards P, then for corresponding amounts of information, $\lim_{n\to\infty} \inf I^n(\mathsf{A}, \mathsf{L}) = I(\mathsf{A}, \mathsf{L})$.

Now let S be a σ-algebra and let all distributions $P(A)$ be σ-additive. Let X denote "a measurable space" and let $\xi^*(A)$ denote an event that $\xi \varepsilon A$. One may consider a random element $\xi \varepsilon X$ as a homomorphic mapping $\xi^*(A) = B$ of a Boolean algebra S_X into the Boolean algebra S. The subalgebra of S obtained under the mapping ξ^* of S_X will be denoted by S_ξ. It is natural to assume $I(\xi, \eta) = I(S_\xi, S_\eta)$.

Let $X \times Y$ be a measurable space. The condition $(\xi, \eta)^*(A \times B) = \xi^*(A)\xi^*(B)$ defines the homomorphism of the algebra $S_{X\times Y}$ into S. This defines in turn a pair (ξ, η) as a random element of the space $X \times Y$. The formula $P_\xi(A) = P[\xi^*(A)]$ gives a measure in X. If a measure on the product space $X \times Y$ is also given, then according to the Radom-Nicodym theorem $P_{\xi\eta}(C) = \iint_C a(x, y)\, dP_\xi\, dP_\eta + S(C)$ where the measure S is singular with respect to $P_\xi \times P_\eta$.

Theorem 4: $I(\xi, \eta)$ is finite only if $S(C) = 0$ and we have then $I(\xi, \eta) = \iint_{X\times Y} a(x, y) \log a(x, y)\, dP_\xi\, dP_\eta$.

Suppose now that X and Y are complete metric spaces. *Theorem 5*: if for random elements $\xi \varepsilon X$ and $\eta \varepsilon Y$ the distributions $P_{\xi,\eta}^{(n)}$ converge weakly towards the distribution $P(\xi, \eta)$, then for corresponding amounts of information, $\lim_{n\to\infty} \inf I^n(\xi, \eta) \geq I(\xi, \eta)$.

Error Statistics Utilizing the Code Translation Data System over Various Media—E. J. Hofmann (in English). Lincoln Lab., M.I.T., Lexington, Mass., Rept. No. 25G0026; March 28, 1961.)

In order to obtain the comparative characteristics of the Lincoln Laboratory code translation data system (CTDS) on various digital data transmission media, a series of tests was run from October, 1959, through August, 1960, over six private line loop circuits. Three of these were K-carrier circuits; two were microwave TD-2 (L-3 carrier) circuits; and one was an H-44 cable.

It is shown that the average error rate for both the K-carrier circuits and the H-44 cable is about 1 bit error in 10^6, and a magnitude higher for the microwave circuits. The number of bit errors per word error varies from below 2 to over 6, the higher number being predominant. The temporal distribution of errors is shown to be of a burst nature interspersed with long quiet intervals.

A Note on Optimum Linear Multivariable Filters—R. J. Kavanagh (in English). (IEE Monograph No. 439 M; April, 1961.)

A multivariable system is described by a matrix of power spectra which specifies the spectral distribution of signals from each variable and the correlations between variables. One problem is to specify a physically realizable filter which will transform this set of spectra into another set in which outputs from individual variables are both uncorrelated and have white-noise spectral distributions. An explicit solution can be found in three stages. First, a multiplier $A(p)$ will be found which transforms the original matrix into a diagonal matrix corresponding to the elimination of cross-correlations. Then the matrix is to be further modified by a multplier $B(p)$ so that the individual spectra will be of white-noise form. The product $B(p)A(p)$ will not in general correspond to the transfer function of a physically realizable system, so it is to be multiplied by $C(p)$ which has zeros placed so as to cancel all right-half p-plane poles of $B(p)A(p)$. The final form $D(p) = C(p)B(p)A(p)$ is physically realizable, but usually includes nonminimum-phase elements. The converse problem, of converting a set of incoherent white-noise signals into a set having a desired matrix of power spectra, can be accomplished by the same technique. A two-variable example is given.

On an Approximation to the Correlation Function of Passive Disturbances—J. Kulikowski (in Polish). (*Prace Przemyslowego Instytutu Telekomunikacji*, no. 31, pp. 29–32; 1960.)

The Fourier spectrum of an autocorrelation function may be approximated by means of a rational function. This leads in turn to the possibility of a useful approximation of the autocorrelation function itself.

A Description of the Statistical Properties of Phase by Means of Periodically-Normal Functions—J. Kulikowski (in Polish). (*Prace Przemyslowego Instytutu Telekomunikacji*, no. 31, pp. 33–37; 1960.)

The density functions of phase are in many cases too complicated to be immediately used in practical computations. It is shown that the approximations to these densities by means of periodically-normal functions leads to substantial simplifications.

Synthesis and Perception of Japanese Fricative Sounds—K. Nakata and Y. Kadokawa (in Japanese). (*J. Inst. Elec. Commun. Engrs. (Japan)*, vol. 44, pp. 221–227; February, 1961.)

Experiments on the synthesis of Japanese fricative sounds are carried out and their results are given in this article. 1) For unvoiced fricative sounds, the frequency characteristics of the noise spectra,

the relative intensity of the consonantal part by noise to the following vowel by buzz, and the frequency locus of the second formant of their transitional parts are studied. 2) For voiced fricative sounds, the transitional characteristics of the first formant and the time relationship of hiss and buzz sources are considered as the factors of voicing, and their effects on voicing of fricative sounds are studied.

The features of Japanese fricative sounds are discussed, comparing the results of their synthesis to English fricative sounds. A new method of synthesis is applied for the [*h*] sound which has a higher audibility than by any other method; the characteristics of this method are given in detail.

A Consideration of *P*-Nary Codes—S. Noguchi, *et al.* (in Japanese). (*J. Inst. Elec. Commun. Engrs. (Japan)*, vol. 44, pp. 205–211; February, 1961.)

Fundamental problems of *P*-nary codes are considered here. By introducing measures in the code space, a concept of distance between two codes is defined and the relationship between the distance and its physical meanings, which is represented here by transition probabilities, is derived.

A set of codes with distance greater than a specified value is composed in the code space spanned by the distance of codes from the viewpoint of the theory of groups, and the number of the codes is calculated. When $P = 2$, this number coincides with that of Muller.

Small Signal Detection through Binomial Sequential Analysis—C. Nuese (in English). (Navy Electronics Lab., San Diego, Calif., Rept. No. 766; March 25, 1957.)

A method for radar detection by sequential analysis through Bernoulli trials of Rayleigh-distributed data is described. This method requires less delicate circuits than methods using the complete data, at a cost of increasing the average sample size by 55 per cent. An optimum voltage quantizing level is computed and curves are presented showing the average sample sizes required. A digital computer is proposed to perform this test.

On Certain Problems Concerning the Detection of Weak Signals—B. Picinbono (in French). (*Ann. des Telecommun.*, vol. 16, pp. 2–27; January-February, 1961.)

In the first part, the author describes the realization of an experimental method designed to measure the one-dimensional probability density of a stationary noise voltage. This method is used to study probability curves which are difficult to obtain by computation, as well as to show experimentally the tendency toward a Gaussian law caused by selective filtering of non-Gaussian noise. This problem is studied from a theoretical viewpoint in the second part, where a certain number of necessary and sufficient conditions are given. Finally, the last part is devoted to the study of the effect of clipping on the detection of weak signals by correlation techniques.

Signal Processing in Radar Astronomy-Communication via Fluctuating Multipath Media—R. Price and P. E. Green, Jr. (in English). Lincoln Lab., (M.I.T., Lexington, Mass., Tech. Rept. No. 234; October 6, 1960.)

Measurement of, and effective detection of or communication via, a propagation medium (such as is encountered in reflection from a deep fluctuating radar target, or in scatter-communications or underwater sound) that has significant multipath spread and/or fluctuation rate is studied in terms of the *scattering function* which describes how the medium redistributes the transmitted power in time and frequency.

In measurement, matched-filter detection is employed, the *ambiguity function* (Woodward-Ville time-and-frequency correlation function) of the transmission giving a "window" through which the scattering function is observed. The scattering function is related to physical and geometrical properties of deep fluctuating radar targets, and the extent of its time-frequency spread is shown to affect its measurability.

The detection-communication analysis assumes that at the receiver input there is sufficient white noise that a low SNR prevails within the bandwidth of the signal that arrives via the propagating medium. Optimum detection, and detection using a simple filter, are considered, the former having several alternate circuit realizations. For a medium having both multipath spread and fluctuation, a *RAKE radiometer* receiver is derived for the optimum detector, which is an iteration of a *weighted radiometer*; the latter structure is an optimum detector for a single-path fluctuating medium, and combines radar sweep integration with radiometry. Optimum fixed-energy transmissions are found for certain scattering functions, and are shown to be "low-TW" and to have a duration that is the geometric mean of the multipath spread and the fluctuation period. For media of small time-frequency spread the simple filter-detector works nearly as well as the optimum detector, but the effectiveness of both decreases as the spread increases, the filter-detector becoming steadily worse relative to the optimum detector.

Some actual radar experiments on the Sun and Venus, relating to the preceding analysis, are briefly described.

Semi-Coherent Detection—I. S. Reed (in English). (The RAND Corp., Santa Monica, Calif., Rept. No. P-2106; September 19, 1960.)

In this paper a method is presented for encoding "and transmitting" messages over an RF channel which could be shifted in frequency by a Doppler frequency. The receiver-detection process described herein avoids the necessity of a precise knowledge of the Doppler shift; it is only necessary to know the frequency band in which the shift lies. Thus, with this process, called semicoherent detection, the need for either a filter bank of matched filters or a Doppler tracking filter is avoided. In view of the receiver simplicity this communications technique could well have application in communicating to satellites and space probes. Its detection sensitivity is comparable to square-law detection over the same time-bandwidth product.

The Design of an "Error-Free" Data Transmission System for Telephone Circuits—B. Reiffen, *et al.* (in English). Lincoln Lab., M.I.T., Lexington, Mass., Rept. No. 25G-0029; December 22, 1960.)

Recent experimental results verify that easily implemented codes can detect essentially all errors occurring in digital data sent over toll grade telephone circuits. A two-way communications system is described which uses these codes to detect the occurrence of errors and requests a retransmission of any data in error. The feedback logic is described in detail and the buffer required to adapt the system to various data sources is discussed. Extrapolated experimental results indicate that the system will deliver data in each direction at an average rate of approximately 7/8 the modem bit rate with a mean time to error of several hundreds of years. The philosophy that guided the design of this system can be applied to other media where high noise bursts or low signal levels occur infrequently.

Sequential Detection Statistics—R. E. Simmons and R. A. Worley (in English). (Navy Electronics Lab., San Diego, Calif., Rept. No. 963; April 4, 1960.)

The sequential probability ratio test (detection process) was investigated under the assumption of Rayleigh distributed signal and noise; both an optimum sequential test and a sequential test on binomially quantized observations were considered. For the case of a small sample size, methods are given for finding the statistical moments and distribution of sample size (detection time) and the error probabilities (false alarm and miss probabilities). Results of three methods of prediction of detector preformance are compared. Methods are indicated for predicting the behavior of a sequential test modified by forced termination an a sequential test performed with the target present during only part of the test.

Biological and Artificial Intelligence—D. I. Sweitzer (in English). (Jet Propulsion Lab., Pasadena, Calif., Literature Search 254; December, 1960.)

During recent years an interest has been generated in the possibility of constructing a machine to simulate throught processes, even to the extent of making independent decisions on the basis of sensory data coupled with programmed or learned experience. A compilation has been made of references dealing with biological intelligence, including perception, learning and decision making, and also the simulation of these abilities. The following sources were consulted: Armed Services Technical Information Agency, *Technical Abstract Bulletins*, through March, 1960; *Aero/Space Engineering*, 1958–1959; *Computers and Automation*, 1957–February, 1960; IRE Proceedings, 1959; *Journal of Symbolic Logic*, 1936–1941, 1958–June, 1959; *Psychological Abstracts*, January, 1954–October, 1959; JPL Library Additions and files; and miscellaneous periodicals.

The material is divided into eighteen sections, as follows: feasibility of simulating thought processes by machine; general automata; models and theories of general thinking processes and behavior; intelligence testing; models and theories of nerve transmission; electrical study of the brain and its functions; physiology and anatomy of the brain; machine perception-mapping; the perceptron; automatic translators; models and theories of perception; machine learning and memory; models and theories of learning; models and theories of memory or recall; decision making by machine; models and theories of decision making; information theory; and man-computer symbiosis. An author index is also given.

The Principle of Causality and the Second Principle of Thermodynamics—J. P. Terletsky (in French). (*J. de Physique et le Radium*, vol. 21, pp. 681–684; October, 1960.)

It is shown that the most reasonable interpretation of the causality principle is to consider it as a consequence of the second principle of thermodynamics. It is considered, according to information theory, that a signal is a localized perturbation, transporting negentropy. In this case, the second principle and the invariance interval of the universe prohibit signals faster than light. However, localized perturbations, transporting energy faster than light, can still exist, if they do not transport negentropy, that is, if they are statistical fluctuations. Thus, particles of imaginary mass, moving with greater speed than light, can be admitted as physical realities, but the process of emission or absorption can only have the characteristics of a fluctuation, and arise without any systematical change of the entropy of the emitting or absorbing body.

The following papers were published singly by the Professional Group on Information Theory (I) and the Professional Group on Automata and Automatic Control (A) of the Institute of Electrical Communication Engineers of Japan, 2–8, Fujimicho, Chiyodaku, Tokyo, Japan. All are in Japanese; English abstracts are given when available.

Information Recognition and Connection Matrices (I; February 24, 1961)—H. Enomoto.

When information symbols belonging to an information source are transmitted to a receiver, transitions between symbols are frequently introduced by various disturbances.The statistical nature of the transition relation can be represented by a transition probability matrix. In this paper, the topological structure which fixes the transition relation between information symbols is discussed by using a multihole torus and a transformation group. The common properties of some of the information symbols are in close relation with a closed path circulating around some holes of torus. Because the normal subgroup of the transformation group has many interesting characteristics, a systematic recognition method is obtained.

The Theory of the Pattern Recognition (I; March 31, 1961)—N. Honda.

In this paper a mathematical method of simplifying pattern recognition is described. It is assumed that by suitable treatments, patterns may be represented in an n-dimensional space Γ whose coordinates have only two values, 0 or 1. We have a set of conceptional letters and each conceptional letter corresponds to a point-set in the Γ space. We assume that these sets of points are given in advance. If all pairs of sets in the Γ space which correspond to different conceptional letters have no common subset, the space is said to be separable.

When we receive a pattern, the process of the recognition is to decide on the conceptional letter corresponding to the received pattern, using the given knowledge on sets of patterns. When the dimension n of the Γ space is large, the above process is troublesome, and it is desired to reduce the dimensionality of the space. In general, it is possible to map the Γ space into another space Γ', conserving separability, where the dimensionality of the latter space is smaller than that of the former. In this paper, the mapping functions are restricted to the linear functions of module 2. The conditions and methods of obtaining the optimal mapping functions are considered.

Application of Miyakawa's Multidimensional Sampling Theorem, III (I; February 24, 1961)—K. Sasakawa.

On the Periodicity of the Time-Envelope of Unharmonic Tones in Pitch Sensation (I; March 31, 1961)—T. Sugimoto.

The pitch of complex tones such as vowels is predicted to be in close correlation with the time-envelope pattern of the waves. This is verified by experiments on pitch and periodicity, using unharmonic tones. Two kinds of periodicity are found in the envelope of the unharmonic tones. One, which appears more often than the other, is found to be coincident with pitch sensation. In this paper, this interesting character of the periodicity is derived by theoretically calculating the periodicty of the time-envelope of simple unharmonic waves.

Mechanical Abstracting (A; March 16, 1961)—R. Tatenuma and N. Sugiura.

Extraction of Pitch Signals in Voice (I; March 31, 1961)—H. Wakabayashi.

Pitch frequency plays an important role in various voice processings. In any case, pitch frequency must be extracted from the uttered voice at the outset. This paper discusses the general idea of pitch extraction and presents two actual examples. Properties of the voice pitch and relations between the pitch and the waveform, which must be clarified beforehand, are also considered.

On All-Purpose Turing Machines with Minimum Size (A; February 16, 1961)—S. Watanabe.

The following papers appear in Volume 1 of the "Proceedings of the Symposium on Decision Theory and Applications to Electronic Equipment Development (May 10–11, 1960)," held at Rome Air Development Center, Griffiss Air Force Base, N. Y. The volume is published as RADC Rept. No. TR-60-70A and is obtainable at the Office of Technical Services, U. S. Dept. of Commerce, Washington, D. C. All papers are in English; authors' affiliations are given below.

An Introduction to Bayes Decision Procedures—N. Abramson (Stanford University, Calif.).

An introduction to statistical decision theory, viewed as an extension of game theory, is given. The two main differences between a decision theory problem (or a statistical game and an ordinary game) are pointed out. From these considerations the three basic components of decision problems are obtained and discussed. These components are 1) the *a priori* information, 2) the decision criterion, and 3) the experiment. Examples are given to illustrate the methods of decision theory and to emphasize the importance of the three basic components. Changes in the decision problem produced by varying each of these three components are also illustrated.

Signal Detection by Adaptive Filters—E. M. Glaser (Johns Hopkins University, Baltimore, Md.).

Most of the work in communication engineering concerned with weak signal detection has been devoted to the study of the synthesis and performance of optimum, time-invariant filters and detectors, that is, to detection systems whose structure is fixed in a configuration which is optimum for the particular signal or class of signals that is to be received. The nature of the signal and the various statistical properties of the interfering noise need to be known to the designer before this class of detection system can be expected to yield any degree of satisfactory performance. The designer's restriction to an invariant system is one of the reasons for this being so. Increasing attention is now being given to detection systems which are able to adapt their structure so as to be optimum for the particular detection problem of the moment.

This paper describes a form of adaptive detection system suitable for the reception of a pulse signal whose waveform is fixed but is unknown at the receiver. The system is one which functions initially as an incoherent detector and, as it receives more and more pulses from a particular source, modifies its structure and optimizes its detection performance with respect to this signal.

The Choice of Estimation Procedure and Loss Function for Radar System Synthesis—J. E. Keigler (Radio Corp. of America, Princeton, N. J.).

In recent years, much effort has been devoted to applying the techniques of statistical decision theory to the radar situation, particularly to the problem of determining whether or not a target is present, that is, the detection problem. The more interesting and difficult problem of inferring the properties of the target from the noise-corrupted returned signal, termed parameter estimation in the language of decision theory, is now receiving its share of attention. One method of estimation, the Bayes procedure utilizing the bounded absolute error loss function, is presented in this paper as the most appropriate for the majority of radar situations.

The Search and Detection Efficiency of Surveillance and Communications Devices Using Sequential Probability Ratio Analysis—G. W. Preston (General Atronics Corps., Bala Cynwyd, Pa.)

The present study was undertaken to determine the logical design—and detailed block diagram form—of the probability ratio sequential search radar and to compute its performance gain in effective transmitter power over nonsequential radars which are otherwise identical. In achieving the objectives of this program it proved necessary to extend the theory of Wald and subsequent workers to take account of two important practical factors: 1) group sampling and 2) detection losses. As a consequence of this theoretical work, we are led to a noticeably different form of the detection apparatus, which luckily gives performance appreciably better than the so-called "Wald detector" in which group sampling is not taken into account. When correctly applied, probability ratio sequential analysis characterizes the optimum search radar since it gives the maximum sensitivity for any specified average search rate and maximum average search rate for any specified sensitivity.

A Sequential Multi-Decision Procedure—F. C. Reed (Planning Research Corp.

This paper considers a sequential approach to the multidecision problem which requires the selection of one of a finite number k of hypotheses H_i. By way of introduction, maximum likelihood and minimum average-risk fixed-sample multidecision problems are considered, and Wald's classical sequential probability ratio test for testing a simple hypothesis H_1 against a single alternative H_2 is discussed. After this introduction, a generalized sequential probability ratio is defined and rules set up for the sequential acceptance of one of the k hypotheses H_i. It is shown that when $k = 2$ this procedure reduces to the classical sequential test. The rules for acceptance effect the errors in decision and the expected sample size required for making a decision. By considering the sequential multidecision problem as a generalized random walk in k dimensions, one may construct the Markov matrix associated with the process, and compute the various errors in decision and the expected sample size. An error analysis is only feasible on a large-scale computer, but a numerical example illustrating the use of the technique is presented.

The Decision Problem in Radar—L. S. Schwartz (New York University, N. Y.).

Well-known decision tests for use in the detection of noisy signals in receiving systems depend on *a priori* probabilities and likelihood statistics. Without such knowledge decision thresholds cannot be validly prescribed and error probabilities computed. In the ususal treatment of detection, one assumes that the necessary information is available, thereby avoiding many difficulties of an *a priori* and predictive nature. It is the objective of this paper to examine the problem of detection for various degrees of deficiency in the knowledge considered necessary for use of customary methods and to indicate what can be done by other means.

Inductive probability is shown to have significance for radar from the standpoint of reliability, and a method for implementing inductive systems which function efficiently with unknown and randomly varying channels statistics is presented.

The Resolvability of Point Sources—P. Swerling (The RAND Corp., Santa Monica, Calif.)

The problem investigated is the resolvability, in the presence of noise, of point sources which are separated in azimuth by less than the width of the main lobe of the gain pattern. The probability of correctly resolving two sources is derived for a particular class of decision methods as a function of the strengths of the sources, their angular separation and noise level. Numerical results are presented for a case which corresponds more accurately to optical or infrared devices than to radar.

Book Reviews

Lectures on Communication System Theory—Elie J. Baghdady, Ed. (McGraw-Hill Book Co., Inc., New York, N. Y.; 1961.)

"In the last analysis, radio communications systems are designed by 'seat-of-the-pants' engineering—let no forest of formulae in this or any other chapters of this book suggest otherwise."

This statement, taken from Chapter 2 of the book by D. G. Brennan, is largely true, and the resulting implications are worthy of some examination. This situation certainly merits comment, especially before reviewing a book that does indeed display "a forest of formulae" in its 617 pages and whose title implies that the contents are concerned with communication systems.

In the past two decades, much effort has been expended and some notable progress has been made in an area that some choose to describe as "statistical communication theory." The works of Wiener and Shannon are invariably (and properly) given special mention in discussions of this nature, but there are certainly a large number of other people who have also made significant contributions to the over-all effort. In view of the attention, effort, talent, and results that have been associated with statistical communication theory, one could logically expect to find, at this relatively late date, communications system *engineering* to be heavily influenced by statistical communication *theory*. The least to be expected is that several of the key results from the theory have been adopted by the engineer with beneficial results in practical application. Unfortunately, neither of these events has transpired, nor is there any indication that the theory will influence the practice to any great extent in the near future.

It is often said that good theory and good practice go hand in hand. How then do we account for this apparent void between the engineer and the theorist in communications work? That this void exists, there is little doubt; the problem has been discussed quietly but persistently for several years. If one chooses to listen to extreme positions in this matter, a rather poor opinion of both groups might result. The theorist has claimed that the engineer is a technical reactionary, unable or unwilling to accept or try new approaches, preferring the familiarity and relative safety of proven and generally accepted techniques. The theorist, on the other hand, has been accused of being disinterested in the real challenges which exist, preferring instead to invent problems for which he has or wishes to find solutions. Indeed, more than one engineer, after studying the theoretical literature in communications of recent vintage, has come away convinced that many of these writers are merely using communications as an excuse in order to display one type of mathematical proficiency or another.

If forced to comment on such a debate, this writer would agree with both sides. In spite of the self-congratulatory writings one finds so often these days in both theoretical and applied communications journals, the fact still remains that the communications systems we are building today are the same ones we were building 20 years ago. The only real difference between the systems of today and those of two decades ago is components. Certainly there has been much progress in the theory of communications, but where are the applications? Where are the great ideas that stimulate action instead of mere discussion? Where are the men of revolution such as Marconi, DeForest, and Armstrong?

Perhaps our trouble today is, in part, the very fact that communications people tend to drift either into application or theory. The future may well belong to neither group, but rather to those individuals who are able and willing to master both disciplines. The handbook engineer, while still quite useful, is not likely to provide the spark that will push us ahead. The strict theorist is all too prone toward an involvement with models and mathematics that may be elegant, interesting, and challenging, but still trivial (if not irrelevant) from a basic application point of view.

A refreshing example of the results which may be obtained by combining creative thinking, theory, and engineering may be found in a recent publication[1] by Kotel'nikov. This brilliant Russian scientist certainly knows his theory, but he never becomes obsessed by it. The most impressive thing about Kotel'nikov's book, to this writer, is the fact that the ultimate goal of all work, application, is never once lost from sight. If the Soviets have managed to give this man authority without turning him into an administrator, then we should not be surprised to find ourselves behind in communications as well as in rocketry.

It would be pleasant to report that the book being reviewed continues the good work of Kotel'nikov in bringing theory and practice more closely together. Such is not the case; the book does more to demonstrate the current situation in communications than to alter it. Communication systems will still be designed by "seat-of-the-pants" engineering, the "forest of formulae" to the contrary. This is hardly a criticism but merely a statement of probable fact, for, when viewed as a report on the current status of communications philosophy, the book succeeds rather well. One interesting attempt in the book, to reduce a popular theory to practice, ends in disaster but perhaps teaches a lesson. I refer here to Section III of Chapter 11 by R. M. Lerner. The capacity formula for a noisy channel is used in an attempt to contradict the engineering axiom concerning the exchange of data rate for an improved error rate. Lerner demonstrates an application of the formula by the use of a multisymbol alphabet of optimally-chosen waveforms. Of course, Lerner fails to reach the promised land of arbitrarily small error rate because of the rapidly diminishing return in power gain as the alphabet size becomes large. Admitting the obvious impracticality of continuing such an approach, Lerner then suggests that the symbol number can be limited and that further improvements may be obtained by a digital coding scheme. This certainly is no answer since digital coding, under the conditions assumed, represents merely a less efficient extension of the multisymbol technique already in use. One can only conclude at this point that the old engineering axiom has survived rather well. Now, do we conclude that Lerner has failed or could it be that the theory has failed? Is it not somewhat presumptuous to assume that every theory has a useful application? Has it ever been shown that information theory, for example, has more than a vocabulary relationship to communications? Lerner declines to discuss questions such as these that his work very obviously suggests. We shall follow his wise example and return at once to the business of review.

The subject book edited by Baghdady certainly reflects the wide range of talents which have been attracted to communications work. The contents vary from application of Shannon's information theory in feedback communications systems to an engineering comparison of active and passive satellite relay stations. The book is essentially a collection of papers resulting from the M.I.T. summer program (1959) on "Reliable Long-Range Radio Communication." The 18 authors are recognized authorities in their respective specialities, and the reader can appreciate the effort required to edit the contributed material into the 23 chapters which make up the book. Baghdady's editing work is certainly adequate for publication purposes, and this is probably all the editing needed since most readers will refer to selected chapters rather than treat the book as an entity. For the most part, the chapters can be read individually without need for outside reference.

In addition to editing, Baghdady has contributed a chapter on diversity techniques and one on analog modulation systems. The first of these represents a good treatment of a popular and important subject. Selection, maximal-ratio and equal-gain diversity systems are analyzed and discussed. The accompanying mathematics is adequate and, in addition, practical considerations are given a good exposition. His second chapter on analog modulation systems runs over 100 pages in length and would almost qualify as a book in itself. Linear modulation (AM and derived systems) is considered, but the main effort is reserved for exponential modulation systems and, more specifically, FM. FM, like most other nonlinear modulation techniques, involves rather difficult systems analysis problems. As a consequence, this chapter is not particularly easy to read. Writing on a subject such as this presents a real challenge, since a good measure of engineering judgement is required to supplement

[1] V. A. Kotel'nikov, "The Theory of Optimum Noise Immunity," McGraw-Hill Book Co., Inc., New York, N. Y.; 1959.

direct analysis. It is always interesting to observe how good engineers attack problems for which exact analyses are impossible or impractical. Thus, the reader who is not particularly concerned with FM can study this chapter as a demonstration of engineering method. Those who are concerned with FM problems, especially with regard to some of the new detection techniques and the claimed improvements in threshold level, will want to give this material a careful reading. Baghdady's claims for FM performance have aroused some controversy of late, which gives this chapter added interest. Regardless of the side one chooses to take in this matter, Baghdady must be credited with putting his ideas in print where all may examine them in detail.

W. M. Siebert has contributed one chapter on linear, time-invariant signal processes and one on decision theory. The first of these covers familiar ground, such as linear networks, the sampling theorem, and band-limited, white, Gaussian noise, in a brief but professional manner. The second chapter by Siebert on decision theory represents an excellent exposition of the fundamentals of this approach. This chapter is highly recommended to engineers who are curious about this area of theoretical endeavor, but who have found difficulty in the past in obtaining readable material on the subject. A more general formulation of decision theory is given in a following chapter by W. L. Root. Both Siebert and Root give examples of applications of decision theory to communications problems.

The representation and design of signals are topics covered in two chapters by R. M. Lerner. In much communications work, the form of the transmitted signal is usually predetermined by convention or regulation. The analysis work involved in such cases usually relates to the optimum recovery or decision-making process, given a set of transmitted waveforms. Lerner's work is concerned with the more general problem of determining the transmission waveforms to be used for best performance in a given situation. This is a relatively new approach, and Lerner discusses some interesting problems. Multisymbol systems are treated for the case of a white, Gaussian channel disturbance, and the cases of impulse and other non-Gaussian disturbances are also considered. The non-Gaussian nature of noise found at the lower radio frequencies and on telephone lines presents some interesting systems problems, and a reading of Sections V and VI of Chapter 11 is especially recommended.

The difficulties associated with the use of a time-varying transmission medium is a topic that receives attention in four chapters of the book by W. E. Morrow, Jr., H. Sherman, T. Kailath, and J. M. Wozencraft. For the most part, these writers are aware of the danger of taking mathematical models of the medium too seriously, and suitable cautions to the reader are mentioned. This reviewer feels that such qualification is most important since many media are not only statistical in nature, but are also nonstationary to a discouraging degree.

The currently very popular subject of coding is represented in a chapter by P. Elias who gives some special consideration to erasures or null-zone detection techniques. In another chapter, P. E. Green, Jr., considers feedback communication systems from a strict Shannon sense. Predecision feedback is used and some interesting results are derived concerning the effect of the feedback link on channel capacity and possible system simplifications offered by the feedback link for a given level of performance.

Receiving system noise considerations are well treated by R. P. Rafuse and R. H. Kingston, and an associated chapter by A. Uhlir, Jr., should be welcomed by those interested in a survey of semiconductor applications at microwave frequencies. Another fine bit of survey work may be found in the chapter by I. Pollack concerning the performance criteria of speech systems.

The above review omits some of the content, but the reader should have a good idea of the wide variety of topics covered. Each chapter is followed by a list of pertinent references which appear quite useful.

John P. Costas
13 Edgewood Dr.
Fayetteville, N. Y.

An Introduction to Statistical Communication Theory—David Middleton (McGraw-Hill Book Co., Inc., New York, N. Y.; 1960. 1070 pages + 20 index pages + xix pages + 7 bibliography pages + 32 appendix pages + 9 glossary pages. Illus. 6 1/4 × 9 1/4. $25.00)

Some years back, an English statesman on a tour of the United States visited, among other places, Mt. Rushmore National Monument. He is said to have stared at the four presidential faces, carved out of the mountain, for some time. Finally, he turned to his companions and exclaimed, "Marvelous—but is it art?" We feel that we are in the same sort of predicament in trying to categorize Middleton's mountainous treatise. There have been technical books with more pages than this one (though not many), and there have certainly been technical books covering a wider range of topics (though few have been at such an advanced level). Few technical books, however, can claim to be packed with such a weight of material as this one.

This "introduction" to statistical communication theory has a distinctly nonintroductory set of prerequisites. The preface warns "... little space is devoted to the probabilistic and statistical background, the elements of which the reader is assumed to possess. A knowledge of Fourier- and Laplace-transform methods, contour integration, matrices, simple integral equations, and the usual techniques of advanced calculus courses is also required, along with the elements of circuit theory and the principles of radio, radar, and other types of electronic communications systems."

If the preparations for this extended tour of statistical communication theory are somewhat demanding for an introduction, it must be admitted that almost all the points of interest are covered. Some of the special topics omitted are mentioned in the author's preface. These include feedback communication systems, coding theory, and sequential decision theory.

We feel that the title is a misleading guide in trying to categorize this book. Middleton states three purposes for the book in his preface: 1) "to outline a systematic approach to the design of optimal communication systems ···", 2) to unify earlier work and 3) to provide a text. Let us discuss the contents of the book in view of these purposes.

Just as Mt. Rushmore, Middleton is divided into four parts. The first, "An Introduction to Statistical Communication Theory," is certainly the weakest. This part consists of a good deal of introductory material of an expository nature. The basic concepts involved in random variables, random processes, expectations, linear and nonlinear systems, spectra, correlation, sampling and SNR are treated. Most of this material is covered in a less complete but more illuminating fashion in Davenport and Root. There is an encyclopedic quality to this book which makes itself felt primarily in Part I. Seventy-five interminable pages are spent in various definitions of (omitting some duplicating terms) the autocorrelation function, the autovariance function, the autocorrelation coefficient, the autovariance, the intensity, the intensity spectrum, the average power density, the intensity density, the spectral intensity distribution, and the spectral density for random, deterministic and "mixed" processes. Several types of Wiener-Khintchine theorems are discussed in these seventy-five pages but the important topic of estimation of the spectral density is left for later.

Chapter 4 which deals with sampling, interpolation and random pulse trains provides an unfortunate example of a heavy-handed treatment of a subject which deserves better. This chapter contains a large number of results on the spectra of random pulse trains, many of which we have not seen in print before. It does not, however, provide the reader with the insight demanded in such expository material. Middleton has always been a master at solving problems and it is this talent which he applies in Chapter 4. A book of this sort, however, has a higher responsibility than that of providing answers to questions and of solving problems. It has the responsibility of providing insight to the class of problems treated. It has the responsibility of examining the answer and asking, "why?" This last question is rarely asked and this is a disappointment in Chapter 4, as well as in the rest of the book. The last chapter of Part I presents a 40-page introduction to information theory. Shannon's two fundamental theorems are stated but not proved in this chapter. In general, the material compares unfavorably with roughly equivalent material in Feinstein.

Part II, "Random Noise Processes," starts with a discussion of the normal random process and processes derived from the normal. The book may be said to come alive with Chapter 9, "Processes

Derived From the Normal." Properties of normal narrow-band signals and noise are derived together with those of signals and broad-band normal noise processes. A few results on zero crossings and extrema of a random process are also presented. Many of the results of this well-organized chapter are presented in a large collection of problems which amplify the results in the book.

Chapters 10 and 11 deal with the physical origins of the random processes treated in the rest of the book. The Langevin, Fokker-Planck and Boltzmann equations are examined. Finally, models of thermal, shot and even impulse noise are formulated and various properties of these models are obtained. A great many results from a wide variety of sources are combined into the most readable treatment of noise generation we have yet seen.

Part III, entitled "Applications to Special Systems," is devoted to modulation, demodulation and Wiener theory. Much of Part III is taken from Middleton's earlier papers in modulation and demodulation theory. Chapters 12 and 14 provide an extensive treatment of the second-order statistics of amplitude-modulated and frequency-modulated signals. Chapters 13 and 15 discuss in astounding detail the detecton of such signals in the presence of noise. These four chapters will undoubtedly constitute the standard work on such problems for many years.

Chapter 16 treats some measurement problems, Wiener filters and matched filters. Chapter 17 covers a miscellany of distribution problems, some of which are needed in the last part of the book. The material of Chapter 17, perhaps because it is somewhat apart from the main stream of the book, does make interesting reading; *e.g.*, did you know (page 746) that the distribution of the finite-time estimate of the spectral density of a Gaussian random process becomes a simple exponential as the observation time increases?

Part IV, "A Statistical Theory of Reception," is concerned with the application of statistical decision theory to communication problems. Chapter 18 serves to provide an introduction to statistical decision theory. Little selection of material has been used in this chapter and it seems unnecessarily involved for the purposes to which it is put. In the next two chapters, Middleton considers first the general theory of binary detection systems and then a wide variety of examples of such systems. The famous Middleton threshold binary detection expansion rears its ugly head at the end of Chapter 19 and most of Chapter 20 is, unfortunately, based on this material.

The most basic problem in the application of statistical decision theory to the detection of continuous signals is without doubt the detection of a known signal in Gaussian noise. Not only is this problem important in its own right but it may be used as the starting point of almost all other signal detection problems for which a solution is known. One need only average the likelihood ratio for the known signal over the ensemble of possible signals in order to obtain these results. Middleton's chapter on examples of binary detection systems would have made more sense if he had not deferred treatment of this basic problem until he had given more than fifty pages of examples based on his threshold expansion. It seems to me that the threshold expansion dominated Chapter 19 and 20 to an extent which is completely unwarranted.

Chapter 21 presents a number of general results in estimation theory applied to signal extraction. A brief and unsatisfactory chapter deals with information measures and reception and a final chapter mentioning some "Generalizations and Extensions" concludes the main body of the book—but that is not all!

After the 23 chapters outlined above, Middleton has provided 70 pages consisting of: 1) an appendix on "Special Functions and Integrals," 2) an appendix on "Solutions of Selected Integral Equations," 3) "Supplementary References and Bibliography," 4) an *eight page* "Glossary of Principle Symbols," 5) "A Name Index," and 6) a remarkably complete and well-organized "Subject Index."

Let us return now to the three aims of this book as outlined in the preface. A systematic approach to the design of optimal communication systems was outlined by Middleton and Van Meter in 1955. The present book does provide a more unified treatment of much of Middleton's early work. This is especially true of his 1946 and 1950 papers on modulation theory and of his pioneering work in the applications of statistical decision theory. The more detailed treatment and the many examples in the book, however, can hardly be said to add to the original systematic approach. The value of this book as a text is, we feel, almost nil. The difficulties previously mentioned together with the added difficulty of learning fundamental ideas from a book which needs an eight page "Glossary of Principle Symbols" do not recommend this book to the novice.

The value of this book is in its scholarship and not in its pedagogy. The examples given, the problems provided, and the complete references at the end of each chapter make "An Introduction to Statistical Communication Theory" a necessity for the research worker in statistical communication theory.

Norman Abramson
Electrical Engrg. Dept.
Stanford University
Stanford, Calif.

Korrelationselectronik (Correlation Electronics)—F. H. Lange (VEB Verlag Technik, Berlin, Germany; 1959. 320 pages + 8 index pages + 15 bibliography pages. Illus. 6 × 8 1/2. In German.)

This book has two main parts, the first of which lays the foundations of "correlation analysis." The first two chapters summarize the mathematical definitions and principal methods involved in the analysis of correlation coefficients, correlation functions, amplitude spectra and power spectra. Although no new material seems to be presented, this reviewer enjoyed reading the 80-page summary for two reasons: first, the mathematics is written in terms familiar to the modern radio engineer. The author seems to sympathize with readers (like this reviewer), who prefer not to plow through page after page of mathematical formalism when results can be obtained in a straightforward manner, perhaps at the expense of a little less generality. Second, each mathematical result is accompanied by a clear description of its meaning and significance. Thus, the average engineer will understand the highlights of the first two chapters without difficulty. In this reviewer's opinion, the author is less successful in his attempt to establish within the framework of general communication theory certain boundaries between "correlation" analysis, information theory, statistics, spectral theory, signal theory, etc. The third chapter is devoted to the problem of instrumenting correlators and devices for the measurement of correlation functions. Low-pass as well as band-pass devices are treated. Also included is an extensive discussion of the errors which occur in correlators as a consequence of imperfections in multiplier characteristics.

The second part of the book deals with the application of the "correlation analysis" in the field of communications and contains three chapters. The fourth chapter examines the extent to which the correlation function characterizes signals generated by information sources. The concept of the correlation interval is introduced, and is followed by a discussion of the autocorrelation function of speech sounds. Two-dimensional autocorrelation functions are introduced in connection with pictures and printed material. The main theme of the fourth chapter is that in many cases a direct determination of the spectrum is difficult; but the autocorrelation function might be found easily, in which case the Wiener theorem gives the spectrum. The author demonstrates this with the aid of examples in telegraphy, television, and thermal noise. A discussion of the cross-correlation function, correlation matrix, and the rules governing the addition of partially-correlated signals concludes the fourth chapter. The fifth chapter is concerned with the correlation analysis of linear and nonlinear transmission systems. The cross-correlation function between input and output of linear networks is derived. The mean-square error of linear servo systems is treated as an illustrative example. Diversity reception in the presence of fading with partially correlated paths is treated next. Two nonlinear problems have been selected by the author, noise through nonlinear four-terminal networks and various forms of modulators. The last chapter describes various types of correlation receivers. The theory of autocorrelation detectors, the problem of detecting periodic signals buried in noise, and an exposition of the matched filter concept open this chapter. Then the author uses the following examples to illustrate the application of cross-correlation detectors: the RAKE teletype system, the pattern recognition of decimal digits and finally the post-detection interferometer which has been used by radio astronomers.

The list of references contains 335 entries covering the time period up to the middle of 1958.

The author succeeds in his main purpose. This book is a well-written summary of the significance of the correlation function as

a powerful tool in the study of communication systems and as a basis for the design of various devices. The book brings together material which is widely scattered in the literature, and it does not require a high degree of mathematical sophistication on the part of the reader. The author did not suceed, however, in convincing this reviewer that "correlation electronics" represents a separable entity within the framework of general communication theory.

Siegfried H. Reiger
RAND Corp.
Santa Monica, Calif.

Sequential Decoding—J. M. Wozencraft and B. Reiffen (The Technology Press of the Massachusetts Institute of Technology and John Wiley and Sons, Inc., New York, N. Y.; 1961. v + 74 pages. $3.75)

This book reports on important research carried out at M.I.T. and Lincoln Lab. during the past four years in the area of error-correcting codes. It is based on the doctoral theses of the authors and on work by M. Horstein and R. G. Gallagher. It presents a new and possibly promising approach to the problem of error control that is basically different from the block code scheme which has dominated the intensive research on error-correcting codes to be found in the literature in recent years. We first list the contents of the book, then describe in more detail the results presented, make some passing comments, and finally return to a critique of the book.

Sequential Decoding begins with a seven-page chapter discussing the general role of coding in communication as viewed by Information Theory. Chapter 2 presents an excellent short summary of certain known general results concerning the use of block codes on the binary symmetric channel. Bounds on the asymptotic rate of decrease of error probability are derived. The main theoretical results for the new error control scheme as applied to the binary symmetric channel are presented in Chapters 3 and 4. The former treats the reception procedure; the latter deals with the encoding method. Chapter 5 contains the interesting experimental results of the operation of a binary communication system using the sequential error control method as simulated on a large-scale computer. Chapter 6 presents most briefly some extensions of the scheme to more general channels. An appendix containing some bounds on the probability distribution for the sum of identically distributed independent random variables concludes the text.

As little on sequential decoding has previously appeared in the readily available literature, a few technical words about the scheme are in order here. In the strict confines of a book review this description must necessarily be incomplete. We give only the gist of the idea and point out how it differs from the block code approach.

A message source produces a stream of symbols suitable for transmission over a discrete memoryless noisy channel. In the block code scheme first described by Shannon, this sequence of message symbols is segmented into disjoint adjacent blocks, each block being k symbols in length. A fixed encoding dictionary translates each such block into a sequence of $n > k$ symbols which are then transmitted over the channel. The received stream of channel symbols is segmented into disjoint adjacent blocks of n symbols and a fixed decoding dictionary translates each received block into a block of k message symbols. Successive blocks of digits are encoded and decoded independently; the dictionaries remain fixed. Shannon's famous coding theorem asserts that (under certain restrictions) the probability of error in the decoded symbols can be made as small as desired by making n and k large while maintaining $R = k/n$ fixed and by choosing suitable dictionaries for encoding and decoding.

In the scheme described by Wozencraft and Reiffen, encoding is accomplished by a "convolution code of constraint length n." If the rate to be maintained is $R = 1/n_0$ with n_0 an integer, then n must be chosen as an integral multiple of n_0, say, $n = kn_0$. As each message symbol is presented to this convolutional encoder, it produces n_0 symbols for transmission on the channel. The n_0 symbols chosen depend in a stationary way on the last k message digits presented to the encoder. The convolution encoder can be thought of as a window k symbols wide that is slid along the message symbol stream. As the window is moved one symbol upstream, n_0 channel symbols are produced. These channel symbols are a fixed function of the message symbols appearing in the window.

The sequential decoder can also be thought of as a window—one that is slid along the stream of received symbols. This time the window is $n = kn_0$ symbols wide; it is moved along the stream not symbol by symbol but in steps of n_0 symbols at a time. At each step of the window, a decoded message symbol is produced by the decoder. This decoded message symbol is the result of a computational procedure P performed on the n received symbols available in the window at the time. The procedure P itself does not change (in the simplest scheme described) as the decoder window is slid n_0 symbols upstream. Input data to P, however, are the previous k-1 decoded message symbols as well as the n received symbols available in the window. It turns out then that the amount, N, of computation (measured in an appropriate way) done under P for a fixed position of the decoding window is a random variable that depends upon the entire past history of noise symbols on the channel and on the message symbols previously presented for transmission. Details of P are too complicated to permit their description here.

In Chapter 4, the authors show how to generate convolution codes for arbitrary n_0 and k. The codes are easily instrumented. In Chapter 3, they present proofs for the following statements for the case of transmission over the binary symmetric channel:

> 1) Under certain restrictions on the rate R, ($R <$ half channel capacity is sufficient), there exist sequences of convolution codes of increasing constraint length n such that by using the sequential decoding procedure P the error probability of a decoded message symbol can be made vanishingly small *given that the preceding k-1 decoded message symbols have been decoded correctly.* Indeed, this error probability can be made to decrease exponentially with n. Note that the italicized condition makes this statement weaker than the corresponding result known for block codes. 2) The expected value of the number N of computations per message digit performed under P grows less rapidly than some small power of n. (Complete proof of this statement is not given in the book. Rather, proof is presented that a certain subset of the operations in P are so bounded, and the reader is referred to Reiffen's thesis for proof that the remaining set is similarly bounded.)

At first hand, statement 2) above seems in marked contrast to the situation found with block codes where *dictionary size* grows exponentially with n. But this raises a sort of philosophical point. In what sense is it fair to compare *dictionary size* with the number of *computations* made by the receiver per decoded digit? For parity-check block codes the actual *computation* done per block of n received symbol is: 1) the evaluation of n-k linear parity checks, 2) search in a dictionary of size 2^{n-k} for a given word. If one regards the dictionary as permanently wired into the receiver by its maker, then 2) requires no *computation* by the receiver at all! Since k decoded message symbols are produced for each received block, the number of *computations* performed by the parity-check block-code receiver per message symbol is proportional to $(n\text{-}k)/k = R$-1, a quantity that does *not* depend on n. This analysis hedges on the fact that to wire in the dictionary may be an enormous job, in fact, with present knowledge of block codes, it becomes prohibitively costly for sufficiently large n, let us say n in the twenties with present computer technology. But then, if the receiver is to remain in constant use for years on end, very large effort on its initial construction may indeed be justified. The receiver may be very "complex," but it does not do much "computing." In one sense, much of the recent work on block parity-check codes can be viewed as an effort to effect the original organization of the decoding dictionary so that it can be wired into the receiver in a particularly simple, orderly way.

Since N is a random variable in the sequential scheme, its variance as well as its mean is of considerable interest. Theoretical bounds on this variance do not seem to be obtainable, but experimental information on this quantity is given in Chapter 5 which reports on simulation studies of a sequential system. The variability of N is not negligible, and the authors suggest modifications of their scheme to reduce this variability. To this reviewer, the results of the simulation were encouraging, but not conclusive. Much work remains to be done in this field, and it is much too early to make firm statements about the ultimate relative practical merits of block and sequential decoding methods.

I found *Sequential Decoding* a very difficult book to read. A large

part of the difficulty is due to the intrinsic complexity and difficulty of the subject. (After the first reading performed for this review there are still many points of proof I do not understand.) Another part of this is no doubt due to intellectual shortcomings of my own. I feel, however, that a third and not entirely negligible part is due to the organization and exposition by the authors. Words are sometimes used carelessly and subtleties in the logical foundation of proofs are treated much too lightly. Chapter 4 should precede Chapter 3, and I strongly advise a reader embarking on this latter chapter to study Fig. 3.3 and the synopsis section on page 45 before setting sail. The book could benefit by expansion in many sections. The point made in Section 4, page 66, is much too important to be relegated to the last page of the text and hidden among briefly treated extensions. One last critical remark. What purpose could the publisher have in mind in omitting all punctuation in formulas in this and other recent books in the Technology Press Monograph Series?

Despite its difficulty, *Sequential Decoding* is a must for any serious worker in the field of error control. The research reported is impressive and significant. For substantial depression of error probability (large n), the method proposed seems at present to be clearly competitive with if not superior to other methods known to me. What method will ultimately prevail is not clear. The field is still changing rapidly.

D. Slepian
Bell Telephone Labs.
Murray Hill, N. J.

Affiliates of the IRE Professional Group on Information Theory

The Affiliate plan, established by the IRE Board of Directors, enables individuals who are not members of IRE, but have an interest in the fields of information transmission, processing and utilization, to join PGIT without the expense of joining the parent IRE body. Admission as an Affiliate merely requires 1) present membership in a professional society listed below, 2) non-membership in the IRE during the past five years, and 3) payment of $8.50 per year. Membership in both IRE and PGIT will cost $14.00. Members of the following societies are approved for affiliate membership:

American Psychological Association
American Statistical Association
The Institute of Mathematical Statistics
American Documentation Institute
American Geophysical Union
American Institute of Physics and its member Societies:
- Acoustical Society of America
- American Physical Society
- American Association of Physics Teachers
- Optical Society of America

American Mathematical Society
American Speech and Hearing Association
Linguistic Society of America
Linguistic Circle of New York
Modern Language Association of American
Society for Industrial and Applied Mathematics
American Association for the Advancement of Science
American Astronomical Society
American Chemical Society
American Management Association
American Meteorological Society
American Rocket Society
American Society of Mechanical Engineers
Association for Computing Machinery
Audio Engineering Society
Institute of the Aeronautical Sciences
Instrument Society of America
Operations Research Society of America
Society of Motion Picture and Television Engineers

How an Individual May Become an Affiliate of PGIT

Fill out the application blank below. Detach the completed application and mail with your check for $8.50 to the Institute of Radio Engineers, 1 East 79 Street, New York 21, N. Y.

The Institute of Radio Engineers, Inc., 1 East 79 Street, New York 21, N. Y.

PROFESSIONAL GROUP ON INFORMATION THEORY—APPLICATION FOR AFFILIATES

To affiliate with PGIT; indicate the societies of which you are a member (see above list), and remit the assessment shown below to IRE Headquarters with this application.

Information requested below MUST be furnished.

Name (Please Print) .....

Mailing Address .....
Street

.....
City Zone State (Country)

Business or Profession .....

I am presently a member of: .....
(name of society)

.....

I have not been a member of the IRE for the past five years.

I enclose $8.50 which is the publications assessment for PGIT.

(Signed) .....

Index to

IRE TRANSACTIONS

ON

INFORMATION THEORY

Vol. IT-7, 1961

IRE Transactions on Information Theory

Index to Volume IT-7, 1961

CONTENTS

Index to Authors

(Listed by month and page.)

Index to Subjects

(Listed by month and page.)

Index to Footnote References

The following is an alphabetical compilation of all footnote and bibliographical references published in PGIT Transactions *papers and correspondence during the year. The month and page where each reference was listed are given in parentheses.*

A

Abramson, N., "The application of 'comparison of experiments' to detection problems," 1958 IRE National Convention Record, pt. 4, pp. 22–26. (Jul 139)
——, "A Class of Systematic Codes for Non-Independent Errors," Electronics Research Lab., Stanford University, Stanford, Calif., Tech. Rept. No. 51; December 30, 1958. (Oct 244) Also in IRE Trans. on Information Theory, vol. IT-5, pp. 150–157; December, 1959. (Jan 53) (Jul 130)
——: See Linden, D. A.
Ackerlind, E.: See Dilworth, R. P.
Aleksandrov, M. S., "Distribution of changes in phase difference for fluctuating random signals and correlated random noise," *Radio Engrg. and Electronics* (English translation of *Radiotekh. i Elektron.*), vol. 5, pp. 360–365; 1960. (Apr 72) (Oct 267)
Amara, R. C., "The Linear Least Squares Synthesis of Continuous and Sampled Data Multivariable Systems," Electronics Labs., Stanford University, Stanford, Calif., Tech. Rept. No. 40; July 28, 1958. (Jul 189)
Aoki, M., "Dynamic Programming and Numerical Experimentation as Applied to Adaptive Control Systems," Dept. of Engrg., University of California, Los Angeles; February, 1960. (Jul 143)
Arens, R., "Complex processes for envelopes of normal noise," IRE Trans. on Information Theory, vol. IT-4, pp. 204–207; September, 1957. (Jan 20) (Apr 78) (Jul 132, 144)
Arman, A.: See Kovasznay, L. S. G.
Arnoult, M. D.: See Attneave, F.
Aroian, L. A., "The probability function of the product of two normally distributed variables," *Ann. Math. Stat.*, vol. 18, pp. 265–271; June, 1947. (Jan 16)
Attneave, F., "Some informal aspects of visual perception," *Physchol. Rev.*, vol. 61, pp. 183–193; 1954. (Apr 102)
——, and Arnoult, M. D., "The quantitative study of shape and pattern perception," *Psychol. Bull.*, vol. 53, pp. 452–471; 1956. (Apr 102)

B

Baghdady, E. J., "Theory of low distortion reproduction of FM signals in linear systems," IRE Trans. on Circuit Theory, vol. CT-5, pp. 202–214; September, 1958. (Jan 26) (Apr 76)
Balakrishnan, A. V., "A note on the sampling principle for continuous signals," IRE Trans. on Information Theory, vol. IT-3, pp. 143–146; June, 1957. (Jul 133)
——, "Estimation and detection theory for multiple stochastic processes," *J. Math. Anal. and Applications*, vol. 1, pp. 1–12; January, 1961. (Jan 52) (Jul 155)
Barrett, J., "Application of the Theory of Functionals to Communication Problems," Cambridge University, Cambridge, England, Engrg. Lab. Rept.; 1955. (Jul 143)
——, and Lampard, D., "An expansion for some second-order probability distributions and its application to noise problems," IRE Trans. on Information Theory, vol. IT-1, pp. 10–15; March, 1955. (Jul 143)
Barthle, R. C.: See Esheleman, V. R.
Bartlett, M. S.: See Wishart, J. S.
Bateman Manuscript Project, "Tables of Integral Transforms," McGraw-Hill Book Co., Inc., New York, N. Y.; 1954. (Jul 151)
——, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953. (Jul 152)
Battin, R.: See Laning, J.
Belevitch, V., "Synthèse des reseaux éléctrique passifs à n paires de bornes de matrices de repartition prédeterminée," *Ann. Telecommun.*, vol. 6, pp. 302–312; November, 1951. (Jul 189)
Bellman, R., and Kalaba, R., "Dynamic programming and adaptive processes: mathematical foundation," IRE Trans. on Automatic Control, vol. AC-5, pp. 5–10; January, 1960. (Jul 143)
—— and ——, "On adaptive control processes," 1959 IRE National Convention Record, pt. 4, pp. 3–11. Reprinted in IRE Trans. on Automatic Control, vol. AC-4, pp. 1–9; November, 1959. (Jul 143)
—— and ——, "On communication processes involving learning and random duration," 1958 IRE National Convention Record, pt. 4, pp. 16–21. (Jul 143)
Bellman, R., and Kalaba, R., "On the role of dynamic programming in statistical communication," IRE Trans. on Information Theory, vol. IT-3, pp. 197–203; September, 1957. (Jul 139)
Bello, P., "Demodulation of a phase modulated noise carrier," IRE Trans. on Information Theory, vol. IT-7, pp. 19–27; January, 1961. (Apr 72)
——: See Higgins, W. Also Kielson, J.
Bendat, J. S., "Exact integral equation solutions and synthesis for a large class of optimum time variable linear filters," IRE Trans. on Information Theory, vol. IT-3, pp. 71–80; March, 1957 (Jul 144) (Oct 215)
——, "Principles and Applications of Random Noise Theory," John Wiley and Sons, Inc., New York, N. Y.; 1958. (Jul 138, 143)
Benedict, T. R., and Sondhi, M. M., "On a property of Wiener filters," Proc. IRE (Correspondence), vol. 45, pp. 1021–1022; July, 1957. (Jul 144)
Bennan, D., "Some Results in the Estimation of Signal Parameters," Electronics Lab., Stanford University, Stanford, Calif., Tech. Rept. No. 10; September 10, 1956. (Oct 251)
Bennett, W. R., "Methods of solving noise problems," Proc. IRE, vol. 44, pp. 609–638; May, 1956. (Oct 265)

Bergen, A. R., "A non-mean-square-error criterion for the synthesis of optimum finite memory sampled-data filters," 1957 IRE National Convention Record, pt. 2, pp. 26–32. (Jul 144)
Bernstein, R. I.: See Miller, K. S.
Beutler, F. J., "Prediction and filtering for random parameter systems," IRE Trans. on Information Theory, vol. IT-4, pp. 166–171; December, 1958. (Jul 144)
Birdsall, T. G.: See Peterson, W. W.
Birkhoff, G., and MacLane, S., "A Survey of Modern Algebra," The MacMillan Co., New York, N. Y., p. 183; 1948. (Jan 41, 46)
Blachman, N. M., "Minimum-cost encoding of information," IRE Trans. on Information Theory, vol. IT-3, pp. 139–149; March, 1954. (Jan 38)
——, "The output signal-to-noise ratio of a power-law device," *J. Appl. Phys.*, vol. 24, pp. 783–785; June, 1953. (Jan 40)
—— and Tukey, J. W., "The Measurement of Power Spectra from the Point of View of Communications Engineering," Dover Publications, Inc., New York, N. Y.; 1959. (Jul 132)
Blackwell, D.: See Girshick, M. A.
——, Breimann, L., and Thomasian, A. J., "Proof of Shannon's transmission theorem for finite-state indecomposable channels," *Ann. Math. Stat.*, vol. 30, pp. 1209–1220; December, 1958. (Jul 130)
——, *et al.*, "The capacity of a class of channels," *Ann. Math. Stat.*, vol. 30, pp. 1229–1241; December, 1959. (Jul 130)
Blasbalg, H., "Experimental results in sequential detection," IRE Trans. on Information Theory, vol. IT-5, pp. 41–51; June, 1959 (Jul 139)
——, "The relation of sequential filter theory to information theory and its application to the detection of signals in noise by Bernoulli trials," IRE Trans. on Information Theory, vol. IT-3, pp. 122–131; June, 1957. (Jul 139)
——, "The sequential detection of a sine-wave carrier of arbitrary duty ratio in Gaussian noise," IRE Trans. on Information Theory, vol. IT-3, pp. 248–256; December, 1957. (Jul 139, 194)
Bledsoe, W. W., and Browning, I., "Pattern recognition and reading by machine," *Proc. Eastern Joint Computer Conf.*, Boston, Mass., December 1–3, 1959; pp. 225–232. (Jul 135)
Blichfeldt, H. F., "The minimum value of quadratic forms and the closest packing of spheres," *Math. Ann.*, vol. 101, pp. 605–608; 1929. (Oct 270)
Bloom, F. J., *et al.*, "Improvement of binary transmission by null-zone reception," Proc. IRE, vol. 45, pp. 963–975; July, 1957. (Jan 6) (Jul 131, 139)
Blum, M., "Fixed memory least squares filters using recursive methods," IRE Trans. on Information Theory, vol. IT-3, pp. 178–182; September, 1957. (Jul 144)
——, "On the mean square noise power of an optimum linear discrete filter operating on polynomial plus white noise input," IRE Trans. on Information Theory, vol. IT-3, pp. 225–231; December, 1957. (Jul 144)
——, "Recursion formulas for growing memory digital filters," IRE Trans. on Information Theory, vol. IT-4, pp. 24–30; March, 1958. (Jul 144)
Blyth, R.: See Raemer, H. R.
Bodewin, E.. "Matrix Calculus," North-Hall, Amsterdam, Netherlands; 1956. (Oct 233)
Bomba, J. S., "Alpha-numberic character recognition using local operations," *Proc. Eastern Joint Computer Conf.*, Boston, Mass., December 1–3, 1959; pp. 218–224. (Jul 135)
Booton, R. C., Jr., "An optimization theory for time-varying linear systems with non-stationary statistical inputs," Proc. IRE vol. 40, pp. 977–981; August, 1952. (Oct 214)
Bose, A. G., "A Theory of Nonlinear Systems," Research Lab. of Electronics, M.I.T., Cambridge, Mass., Rept. No. 309; May, 1956. (Jul 143)
——, "Nonlinear system characterization and optimization," IRE Trans. on Information Theory, vol. IT-5, pp. 30–40; May, 1959. (Jul 132)
——, "Nonlinear system characterization and optimization," *Trans. 1959 Internatl. Symp. on Circuit and Information Theory.* Reprinted in IRE Trans. on Circuit Theory, vol. CT-6, pp. 30–40; May, 1959. (Jul 143)
Bose, R. C.: See Ray-Chaudhuri, D. K.
Brauer, A., "Limits for the characteristic roofs of matrices," *Duke Math. J.*, vol. 13, pp. 387–395; 1946. (Oct 233)
Breimann, L.: See Blackwell, D.
Brennan, D. G., "On the maximum signal-to-noise ratio realizable from several noisy signals," Proc. IRE (Correspondence), vol 43, p. 1530; October, 1955. (Jul 163)
——, "Linear diversity combining techniques," Proc. IRE, vol. 47, pp. 1075–1102; June, 1952. (Apr 98)
Brillouin, L., "Science and Information Theory," Academic Press, Inc., New York, N. Y.; 1956. (Jan 38)
Brock, R. L.: See Stone, W. M.
Brown, D. T.: See Peterson, W. W.
Brown, J. L., Jr.. "A criterion for the diagonal expansion of a second-order probability distribution in orthogonal polynomials," IRE Trans. on Information Theory (Correspondence), vol. IT-4, p. 172; December, 1958. (Jul 143)
——, "On a cross-correlation property for stationary random processes," IRE Trans. on Information Theory, vol. IT-3, pp. 28–31; March, 1957. (Jul 143)
Brown, W. M., "Some results on noise through circuits," IRE Trans. on Information Theory, vol. IT-5, pp. 217–227; May, 1959. (Jul 132)
Browning, I.: See Bledsoe, W. W.
Brule, J. D., "Improving the approximation to a prescribed time response," IRE Trans. on Circuit Theory, vol. CT-6, pp. 355–361; December, 1959. (Oct 215)
Burge, W. H., "Sorting, trees and measures of order," *Inform. and Control*, vol. 1, pp. 181–197; September, 1958. (Jul 135)
Bussgang, J. J.: See Middleton, D.
—— and Mudgett, W. L., "A note of caution on the square-law approximation to an optimum detector," IRE Trans. on Information Theory (Correspondence), vol. IT-6, pp. 504–505; September, 1960. (Jul 194)
——, Nesbeda, P., and Safran, H., "A unified analysis of range performance of CW, pulse and pulse Doppler radar," Proc. IRE, vol. 47, pp. 1753–1769; October, 1959. (Jul 138)

C

Cahn, C. R., "Performance of digital phase-modulation communication systems," IRE Trans. on Communication Systems, vol. CS-7, pp. 3–6; May, 1959. (Jul 138)
—— and Lawton, J. G., "Comparison of coherent and phase-comparison detection of a four-phase digital signal," Proc. IRE (Correspondence), vol. 47, p. 1662; September, 1959. (Jul 138)
Calabi, L., and Haefeli, H. G., "A class of binary systematic codes correcting errors occurring at random and in bursts," IRE Trans. on Information Theory, vol. IT-5, pp. 79–94; May, 1959. (Jul 130)
Campbell, L. L., "Minimum coefficient rate for stationary random processes," *Inform. and Control*, vol. 3, pp. 360–371; December, 1960. (Jul 193)
——, "On the use of Hermite expensions in noise problems," *SIAM J.*, vol. 5, pp. 244–249; December, 1957. (Jul 132)
Capon, J., "A nonparametric technique for the detection of a constant signal in additive noise," 1959 IRE WESCON Convention Record, pt. 4, pp. 92–103. (Apr 71) (Jul 139)
——, "Optimum coincidence procedures for detecting weak signals in noise," 1960 IRE International Convention Record, pt. 4, pp. 154-166. (Apr 71)
——, "A probabilistic model for run-length coding of pictures." IRE Trans. on Information Theory, vol. IT-5, pp. 157–163; December, 1959. (Jul 135)
Carson, J. R.: See Fry, T. C.
Chang, S. S. L., Harris, B., Hauptschein, A., Hoffmann, D., Morgan, K., and Schwartz, L. S., "Evaluation and Optimization of Digital Communication Systems," New York University, New York, N. Y., 4th Scientific Rept., Contract AF 19(604)-1964; January, 1958. (Oct 222)
Chesler, D. A., "Optimum Nonlinear Filters with Fixed-Output Networks," Research Lab. of Electronics, M.I.T., Cambridge, Mass., Quarterly Prog. Rept., pp. 118–124; July, 1958. (Jul 143)
Chow, C. K., "An optimum character recognition system using decision functions," IRE Trans. on Electronic Computers, vol. EC-6, pp. 247–254; December, 1957. (Jul 135)
Clavier, P. A., "Some applications of the Laurent Schwartz distribution theory to network problems," *Proc. Symp. on Modern Network Synthesis*, Polytechnic Inst., Brooklyn, Brooklyn, N.Y pp. 249–265; April, 1955. (Oct 223)
Craig, C. C., "On the frequency function of xy," *Ann. Math. Stat.*, vol. 7, pp. 1–15; March, 1936. (Jan 16)
Cocke, J., "Lossless symbol coding with nonprimes," IRE Trans. on Information Theory, vol. IT-5, pp. 33–34; March, 1959. (Jan 2)

Cohn, G. I., and Peach, L. C., "Detection of radar signals by direct measurement of their effects on noise statistics," *Proc. Natl. Electronics Conf.*, vol. 14, pp. 821–831; October, 1959. (Jul 138)
Courant, R.: See Hilbert, D.
Cowan, W.: See Galejs, J.
Cramer, H., "Mathematical Methods of Statistics," Princeton University Press, Princeton, N. J.; 1946. (Jan 18) (Apr 62, 71, 98) (Jul 149)
——, "On the theory of stationary processes," *Ann. Math.*, vol. 41, Ser. 2; 1940. (Jul 189)

D

Dantzig, G. B., "Linear Programming and Extensions," to be published. (Jan 38)
Darling, D. A., and Siegert, A. J. F., "A systematic approach to a class of problems in the theory of noise and other random phenomena—Part I," IRE Trans. on Information Theory, vol. IT-3, pp. 32–37; March, 1957. (Jul 132)
Darlington, S., "Linear least-squares smoothing and prediction, with applications," *Bell Sys. Tech. J.*, vol. 37, pp. 1221–1294 September, 1958. (Jul 143)
——, "Nonstationary smoothing and prediction using network theory concepts," *Trans. 1959 Internatl. Symp. on Circuit and Information Theory.* Reprinted in IRE Trans. on Circuit Theory, vol. CT-6, pp. 1–13; May, 1959. (Jul 143)
——, "Nonstationary smoothing and prediction using network theory concepts," IRE Trans. on Information Theory, vol. IT-5, pp. 1–13; May, 1959. Oct 215)
Davenport, W. B., "Signal-to-noise ratios in band-pass limiters," *J. Appl. Phys.*, vol. 24, pp. 720–727; June, 1953. (Jan 39)
——: See Root, W. L.
Davies, I. L.: See Woodward, P. M.
Davis, R. C. "On the detection of pure signals in noise," *J. Appl. Phys.*, vol. 25, pp. 76–82; January, 1954. (Oct 215)
——, "On the theory of prediction of non-stationary stochastic processes," *J. Appl. Phys.*, vol. 23, pp. 1047–1052; September, 1952. (Oct 215)
——, "The detectability of random signals in the presence of noise," IRE Trans. on Information Theory, vol. IT-3, pp. 52–62; March, 1954. (Jul 161)
DeLong, D. F., Jr., "Three-Phase Codes," M.I.T. Lincoln Lab., Lexington, Mass., Group Rept. 47:28; July 24, 1959. (Oct 255)
Deutsch, S., "A note on some statistics concerning typewritten or printed material," IRE Trans. on Information Theory, vol. IT-3, pp. 147–148; June, 1957. (Jul 135)
Dicke, R. H., "The measurement of thermal radiation at microwave frequencies," *Rev. Sci. Instr.*, vol. 17, pp. 268–275; July, 1946. (Jul 163)
Dickinson, W. E., "A character-recognition study," *IBM J. Res. and Dev.*, vol. 4, pp. 335–348; July, 1960. (Jul 135)
Dickson, L. E., "History of the Theory of Numbers," Carnegie Inst. Washington, Washington, D. C.; 1919. (Jan 38)
Dilworth, R. P., and Ackerlind, E., "The analysis of post detection integration systems by Monte Carlo methods," 1957 IRE National Convention Record, pt. 2, pp. 40–47. (Jul 138)
Dimond, T. L., "Devices for reading handwritten characters," *Proc. Eastern Joint Computer Conf.*, Washington, D. C., December, 9–13, 1957; pp. 232–237. (Jul 135)
Dineen, G. P., "Programming pattern recognition," *Proc. Western Joint Computer Conf.*, Los Angeles, Calif., March 1–3, 1955; pp. 94–100. (Apr 102)
Dolph, C. L., and Woodbury, M. A., "On the relation between Green's functions and covariances of certain stochastic processes and its application to unbiased linear prediction," *Trans. Am. Math. Soc.*, vol. 72, pp. 519–550; May, 1952. (Apr 120)
Doob, J. L., "The Brownian movement and stochastic equations," *Ann. Math.*, vol. 43, pp. 351–369; April, 1942. (Jul 150)
——, "Stochastic Processes," John Wiley and Sons, Inc., New York, N. Y., p. 71; 1953. (Jan 20) (Jul 144) (Oct 233, 267)
Doyle, W., "Recognition of sloppy, hand-printed characters," *Proc. Western Joint Computer Conf.*, San Francisco, Calif., May 3–5, 1960; pp. 133–142. (Jul 135)
Drenick, R., "A nonlinear prediction theory," IRE Trans. on Information Theory, vol. IT-4, pp. 146–162; September, 1954. (Oct 257)
Dugundji, J., "Envelopes and pre-envelopes of real waveforms," IRE Trans. on Information Theory, vol. IT-4, pp. 53–57; March, 1958. (Apr 78) (Jul 133, 189)
Dwork, B. M., "Detection of a pulse superimposed on fluctuation noise," Proc. IRE, vol. 38, pp. 771–774; July, 1950. (Jul 189)

E

Elias, P., "Channel capacity without coding," 1957 IRE National Convention Record, pt. 2, p. 49 (Abstract only). Complete text, Research Lab. of Electronics, M.I.T., Cambridge, Mass., Quarterly Progress Rept., pp. 90–93; October 25, 1956. (Jul 139)
——, "Coding for noisy channels," 1955 IRE Convention Record, pt. 4, pp. 37–46. (Jul 130, 171) (Oct 223)
——, "Error-free coding," IRE Trans. on Information Theory vol. IT-4, pp. 29–37; September, 1954. (Jul 130)
——, "Predictive coding," IRE Trans. on Information Theory, vol. IT-1, pp. 16–33; March, 1955. (Apr 99)
——, "Predictive Coding," Harvard University, Cambridge, Mass., Ph.D. dissertation; 1950. (Apr 99)
Elspas, B., "A Radar Based on Statistical Estimation and Resolution Considerations," Electronics Labs., Stanford University, Stanford, Calif., Tech. Rept. No. 361-1; August 1, 1955. (Oct 254)
——, "The theory of autonomous linear sequential networks," IRE Trans. on Circuit Theory, vol. CT-6, pp. 45–60; March, 1959. (Jan 11, 53) (Oct 244) (Jul 130)
Epstein, M. A., "Algebraic decoding for a binary erasure channel," 1958 IRE National Convention Record, pt. 4, pp. 56–69. (Jul 130)
Erdelyi, A.: See Tricomi, F. G.
Eshleman, V. R., Barthle, R. C., and Gallagher, P. B., "Radar echoes from the sun," *Science*, vol. 131, pp. 329–332; February 5, 1960. (Jul 139)
Evey, R. J., "Use of a computer to design character recognition logic," *Proc. Eastern Joint Computer Conf.*, Boston, Mass., December 1–3, 1959; pp. 205–211. (Jul 135)

F

Fano, R. M., Research Lab. for Electronics, M.I.T., Cambridge, Mass., Tech. Rept. No. 65; 1949. (Jan 38)
Feinstein, A., "A new basic theorem of information theory," IRE Trans. on Information Theory, vol. IT-4, pp. 2–23; September 1954. (Jul 130) (Oct 223)
——, "Foundations of Information Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1958. (Jul 130, 171, 194)
——, "On the coding theorem and its converse for finite memory channels," *Inform. and Control*, vol. 2, pp. 25–44; April, 1959. (Jul 130)
Feldman, J., "Equivalence and perpendicularity of Gaussian processes," *Pacific J. Math.*, vol. 8, p. 699; 1958. (Apr 108)
Filipowsky, R., "Recent progress in applying information theory to digital transmission systems," *Trans. AIEE (Commun. and Electronics)*, No. 40, pp. 848–855; January, 1959. (Oct 223)
——: See Scherer, E.
Fire, P., "A Class of Multiple-Error-Correcting Binary Codes for Non-Independent Errors," Sylvania Electronic Systems Rept., presented at AIEE Fall Meeting, Chicago, Ill.; October, 1959. (Jan 53)
Fleckenstein, W. O.: See Michel, W. S.
——, "Flight Display and Flight Control Integration," Research Dept., Melpar, Inc., Watertown, Mass., AF 33(616)-5589; 1959. (Apr 99)
Flores, I., and Grey, L., "Optimization of reference signals for character recognition systems," IRE Trans. on Electronic Computers, vol. EC–9, pp. 54–61; March, 1960. (Jul 135)
—— and Ragonese, F., "A method of synthesizing the wave form generated by a character, printed in magnetic ink, in passing beneath a magnetic reading head," IRE Trans. on Electronic Computers, vol. EC-7, pp. 277–282; December, 1958. (Jul 135)
Fogel, L. J.: See Jagerman, D. L.
Fontaine, A. B.: See Peterson, W. W.
Fox, W. C.: See Peterson, W. W.
Fraser, D. A. S., "Nonparametric Methods in Statistics," John Wiley and Sons, Inc., New York, N. Y.; 1957. (Apr 71)
Freimer, M., "A dynamic programming approach to adaptive control processes," 1959 IRE National Convention Record, pt. 4, pp. 12–17. Reprinted in IRE Trans. on Automatic Control, vol. AC-4, pp. 10–15; November, 1959. (Jul 143)
Friedland, B., "Least squares filtering and prediction of nonstationary sampled data," *Inform. and Control*, vol. 1, pp. 297–313; December, 1958. (Jul 144)

——, "A technique for the analysis of time-varying sampled data systems," *Trans. AIEE*, vol. 73, pt. 2, pp. 407–414; January, 1957. (Jan 53)
——: See Stern, T. E.
Friedman, B., "Lectures on Applications of Distributions," prepared for the Atomic Energy Commission, Brookhaven Natl. Lab.; Summer, 1959. (Oct 223)
——, "Principles and Techniques of Applied Mathematics," John Wiley and Sons, Inc., New York, N. Y., p. 27; 1957. (Oct 215)
Fry, T. C., and Carson, J. R., "Variable frequency electric circuit theory," *Bell Sys. Tech. J.*, vol. 16, pp. 513–540; October, 1937 (Jan 26)

G

Gabor, D., "Theory of communications," *J. IEE*, vol. 93, pt. 111, pp. 429–457; 1946. (Jan 20)
Galejs, J., "Signal-to-noise ratios in smooth limiters," IRE Trans. on Information Theory, vol. IT-5, pp. 79–85; June, 1959. (Jul 139)
——: See Storer, J.
—— and Cowan, W., "Interchannel correlation in a bank of parallel filters," IRE Trans. on Information Theory, vol. IT-5, pp. 106–114; September, 1959. (Jul 139)
Gallagher, P. B.: See Eshelman, V. R.
Gantmacher, F. R., "The Theory of Matrices," Chelsea Publishing Co., New York, N. Y.,; 1959. (Jul 189)
Gel'fand, I. M.: See Yaglom, A. M
George, D. A., "The Prediction of Gaussian-Derived Signals," Research Lab. of Electronics, M.I.T., Cambridge, Mass., Quarterly Prog. Rept.; July, 1958. (Jul 143)
Gilbert, E. N., "A comparison of signalling alphabets," *Bell Sys. Tech. J.*, vol. 31, pp. 504 522; March, 1952. (Oct 223)
—— and Moore, E. F., "Variable length binary encodings," *Bell Sys. Tech. J.*, vol. 38, pp. 933–967; July, 1959. (Jan 38)
Gill, A., "Minimum-scan pattern recognition," IRE Trans. on Information Theory, vol. IT-5, pp. 52–58; June, 1959. (Jul 135)
Girshick, M. A., and Blackwell, D., "Theory of Games and Statistical Decisions," John Wiley and Sons, Inc., New York, N. Y.; 1954. (Oct 246)
Glantz, H. T., "On the recognition of information with a digital computer," *J. ACM*, vol. 4, pp. 178-189; April, 1957. (Jul 135)
Glaser, E. M., "Signal Detection by Adaptive Filters," Rad. Lab., Johns Hopkins University, Baltimore, Md., Tech. Rept AF-75; April, 1960. (Unclassified.) (Apr 98)
—— and Park, J. H., Jr., "On signal parameter estimation," IRE Trans. on Information Theory, vol. IT-4, pp. 173–174; December, 1958. (Jul 143)
Gnedenko, B. V., and Kolmogoroff, A. N., "Limit Distributions of Sums of Independent Random Variables," Addision-Wesley Co., Reading, Mass.; 1954. (Apr 108)
Gohberg, I. C., and Krein, M. G., "Systems of integral equations on a half line with kernels depending on the difference of arguments," *Trans. Am. Math. Soc.*, vol. 14, Ser. 2, pp. 217–287; 1960. (Jul 189)
Golay, M. J. E., "Complementary series," IRE Trans. on Information Theory, vol. IT-7, pp. 82–87; April, 1961. (Oct 273)
——, "Multislit spectrometry," *J. Opt. Soc. Am.*, vol. 39, p. 437; 1949. (Apr 82)
——, "Notes on digital coding," Proc. IRE (Correspondence), vol. 37, p. 657; June, 1949. (Jan 51)
——, "Notes on the penny-weighing problem, lossless symbol coding with nonprimes, etc.," IRE Trans. on Information Theory, vol. IT-4, pp. 103–109; September, 1958. (Jan 2)
——, "Static multislit spectrometry and its application to the panoramic display of infrared spectra," *J. Opt. Soc. Am.*, vol. 41, p. 468; 1951. (Apr 82)
Gold, B., "Machine recognition of hand-sent Morse code," IRE Trans. on Information Theory, vol. IT-5, pp. 17–24; March, 1959. (Jul 135)
Goldman, S., "Frequency Analysis, Modulation and Noise," McGraw-Hill Book Co., Inc., New York, N. Y.; 1948 (Jan 39)
Golomb, S. W., "Sequences with Randomness Properties," Glenn L. Martin Co., Baltimore, Md., Final Rept. on Contract No. SC-54-33611; advance copy dated June 14, 1955. (Jan 12)
Gomory, R. E., "All-Integer Integer Programming Algorithm," IBM Research Center, Yorktown Heights, N. Y., Research Rept. RC-189; 1960. (Jan 38)
——, "An algorithm for integer solutions to linear programs," Princeton-IBM Math. Research Project, Tech. Rept. No. 1; 1958. (Jan 38)
——, "Outline of an algorithm for integer solutions to linear programs," *Bull. Am. Math. Soc.*, vol. 64, pp. 275–278; September, 1958. (Jan 38)
Good, I. J., "Effective sampling rates for signal detection: or can the Gaussian model be salvaged?" *Inform. and Control*, vol. 3, p. 116; June, 1960. (Apr 105)
Graham, R. E., "Communication theory applied to television coding," *Acta Electronica*, vol. 2, 1–2, pp. 333–343; 1957–1958. (Apr 100)
——, "Subjective experiments in visual communication," 1958 IRE National Convention Record, pt. 4, pp. 100–106. (Jul 135)
Green, B. A., Jr., "Radar detection probability with logarithmic detectors," IRE Trans. on Information Theory, vol. IT-4 pp. 50–52; March, 1958. (Jul 138)
Green, J. H., and San Soucie, R. L. "An error-correcting encoder and decoder of high efficiency," Proc. IRE, vol. 46, pp. 1741–1743; October, 1958. (Jul 130)
Green, P. E., Jr., "The output signal-to-noise ratio of correlation detectors," IRE Trans. on Information Theory, vol. IT-3 pp. 10–18; March, 1957. (Jan 41)
Green, P. E.: See Price, R.
Greene, P. H., "Networks for pattern perception," *Proc. Natl. Electronics Conf.*, vol. 15, pp. 357–369; October, 1959. (Jul 135)
Greenias, E. C., and Hill, Y. M., "Considerations in the design of character recognition devices," 1957 IRE National Convention Record, pt. 4, pp. 119–126. (Jul 135)
——, *et al.*, "Design of logic for recognition of printed characters by simulation," *IBM J. Res. and Dev.*, vol. 1, pp. 8–18; January, 1957. (Jul 135)
Grenander, U., "Stochastic processes and statistical inference," *Arkiv. Math.*, vol. 1, pp. 195–277; 1950 (Apr 105) (Oct 233)
——, Pollak, H. O., and Slepian, D., "The distribution of quadratic forms in normal variates: a small sample theory with applications to spectral analysis," *SIAM J.*, vol. 7, pp. 374–401; December, 1959. (Jul 132)
—— and Rosenblatt, M., "Statistical Analysis of Stationary Time Series," John Wiley and Sons, Inc., New York, N. Y.; 1957. (Jul 132)
Grey, L.: See Flores, I.
Grobner, W., and Hofreiter, N., "Integraltafel," Springer-Verlag, Vienna, Austria; 1958. (Jul 153)
—— and ——, "Integraltafel," Springer-Verlag, Vienna, Austria; 1949. (Apr 73)

H

Haefeli, H. G.: See Calabi, L.
Hagelbarger, D. W., "Recurrent codes: easily mechanized, burst correcting, binary codes," *Bell Sys. Tech. J.*, vol. 38–4, pp. 969–984; July, 1959. (Jan 8) (Jul 130)
Halperin, I., "Introduction to the Theory of Distributions," University of Toronto Press, Toronto, Canada; 1952. (Oct 223)
Hammerle, K. J.: See Stone, W. M.
Hamming, R. W., "Error-detecting and error-correcting codes," *Bell Sys. Tech. J.*, vol. 29, pp. 147–160; April, 1950. (Jan 53) (Apr 114) (Jul 130, 171) (Oct 270)
Hardy, G. H., Littlewood, J. E., and Polya, G., "Inequalities," Cambridge University Press, Cambridge, England; 1959. (Jul 149)
Harmon, L. D., "A line-drawing pattern recognizer," *Proc. Western Joint Computer Conf.*, San Francisco, Calif., May 3–5, 1960; pp. 351–364. (Jul 135)
Harris, B.: See Chang, S. S. L. Also Schwartz, L. S.
——, *et al.*, "Binary decision feedback systems for maintaining reliability under conditions of varying field strength," *Proc. Natl. Electronics Conf.*, vol. 13, pp. 126-140; October, 1957. (Jul 139)
Hartmanis, J., "Symbolic analysis of a decomposition of information processing machines," *Inform. and Control*, vol. 3, pp. 154–178; June, 1960. (Jul 135)
Hauptschein, A.: See Chang, S. S. L. Also Schwartz, L. S.
Heasly, C. C., "Some communication aspects of character-sensing systems," *Proc. Western Joint Computer Conf.*, San Francisco, Calif., May 3–5, 1959; pp. 176 -180. (Jul 135)
Helstrom, C. W., "Statistical Theory of Signal Detection," Pergamon Press, Inc., New York, N. Y.; 1960. (Jan 5) (Jul 138) (Oct 233, 245)
——, "The distribution of the number of crossings of a Gaussian

stochastic process," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 232–237; December, 1957. (Jul 132)
——, "The Performance of Communication Channels with Orthogonal Signals," Westinghouse Electric Corp., Research Rept. 412FF471-R1; August 28, 1959. (Jan 5)
——, "The Performance of the Scherer Code in a Quaternary Symmetric Channel," Westinghouse Electric Corp., Research Rept. 412FF471-R2; September 1, 1959. (Jan 7)
——, "The resolution of signals in white Gaussian noise," PROC. IRE, vol. 43, pp. 1111–1118; September, 1955. (Oct 245)
—— and Isley, C. T., "Two notes on a Markoff envelope process," IRE TRANS. ON INFORMATION THEORY (Correspondence), vol. IT-5, pp. 139–140; September, 1959. (Jul 132)
Higgins, W., and Bello, P., "Effect of Limiting on the Probability of Incorrect Dismissal at the Output of an Envelope Detector," Appl. Research Lab., Sylvania Electric Products, Inc., Waltham, Mass., Appl. Research Memo. 163; March, 1959. (Apr 10)
Hilbert, D., "Grundzuge einer allgemeinen Theorie der linearen Integralgleichungen," Nachrichten d.k. Gesellschaft d. Wiss. zu Gottingen, Abschnit 4, 5; 1906. (Oct 215)
—— and Courant, R., "Methods of Mathematical Physics I," Interscience Publishers, Inc., New York, N. Y.; 1953. (Oct 215)
Hildebrand, F. B., "Methods of Applied Mathematics," Prentice-Hall, Inc., Englewood Cliffs, N. J.; 1952. (Jan 46) (Oct 233)
Hill, Y. M.: See Greenias, E. C.
Hoffman, D.: See Chang, S. S. L.
Hofreiter, N.: See Grobner, W.
Hopner, E., "An experimental modulation-demodulation scheme for high speed data transmission," *IBM J. Res. and Dev.*, vol. 3, pp. 74–84; January, 1959. (Jul 138)
Horton, B. M., "Noise modulated distance measuring system," PROC. IRE, vol. 47, pp. 821–828; May, 1959. (Jan 19)
Hseih, H. C., and Leondes, C. T., "On the optimum synthesis of multiple control systems in the Wiener sense," 1959 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 18–31. Reprinted in IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-4, pp. 16–29; November, 1959. (Jul 144)
Huffman, D. A., "A linear circuit viewpoint on error-correcting codes," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 20–28; September, 1956. (Jan 53) (Jul 130)
——, "A method for the construction of minimum redundancy codes," PROC. IRE, vol. 40, pp. 1098–1101; September, 1952. (Jan 38)

I

Isley, C. T.: See Helstrom, C. W.

J

Jagerman, D. L., and Fogel, L. G., "Some general aspects of the sampling theorem," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 139–146; December, 1956. (Jul 130)
Jakowatz, C. V.: See White, G. M.
Janos, W. A., "Optimal filtering of periodic pulse-modulated time series," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 67–74; June, 1959. (Jul 144)
Johnson, V.: See Middleton, D.

K

Kac, M., and Slepian, D., "Large excursions of Gaussian processes," *Ann. Math. Stat.*, vol. 30, pp. 1215–1228; December, 1959. (Jul 151)
Kailath, T., "Correlation detection of signals perturbed by a random channel," IRE TRANS. ON INFORMATION THEORY, vol. IT-6, pp. 361–366; June, 1960. (Jul 138, 161) (Oct 233)
——, "Optimum Diversity Combiners," Research Lab. of Electronics, M.I.T.. Cambridge, Mass., Quarterly Prog. Rept., pp. 198–200; July 15, 1960. (Jul 157)
——, "Optimum receivers for randomly varying channels," *Proc. 4th London Symp. on Information Theory*, C. Cherry, Ed., Butterworth Scientific Publications, Inc., London, England; 1961. (Jul 139) (Oct 233)
Kalaba, R.: See Bellman, R.
Kallianpur, G., "A problem in optimum filtering with finite data," *Ann. Math. Stat.*, vol. 30, pp. 659–669; September, 1959. (Jul 143)
Kalman. R., "A new approach to linear filtering and prediction problems," *J. Basic Engrg.*, vol. 82D, pp. 35–45; March, 1960. (Jul 144)
Kalman, R. E., and Koepcke, R. W., "Optimal synthesis of linear sampling control systems using generalized performance indexes," *Trans. ASME*, vol. 80, pp. 1820–1826; 1958. (Jul 143)
Kamentsky, L. A., "Patterns and character recognition systems—picture processing by nets of neuron-like elements," *Proc. Western Joint Computer Conf.*, San Francisco, Calif., May 3–5, 1959; pp. 304–309. (Jul 135)
Kanter, I., "Some results for the prediction of derivatives of polynomial signals in additive stationary noise," 1959 IRE WESCON CONVENTION RECORD, pt. 4; pp. 87–91. (Jul 143)
——, "The prediction of derivatives of polynomial signals in additive stationary noise," 1958 IRE WESCON CONVENTION RECORD, pt. 4, pp. 131–146. (Jul 143)
Karlin, J. E.: See Pierce, J. R.
Kasami, T., "On the Estimation of Waveform," Inst. Elec. Commun. Engrs. Japan, PGIT Rept.; January, 1960. In Japanese.) (Oct 214)
Kay, I., and Silverman, R. A., "On the uncertainty relation for real signals," *Inform. and Control*, vol. 2, pp. 396–397; December, 1959. (Jul 130)
Kelly, E. J.: See Reed, I. S.
——, Lyons, D. H., and Root, W. L., "The Theory of the Radiometer," Lincoln Lab., M.I.T., Lexington, Mass., Group Rept. No. 47.16; May 2, 1958. (Jul 139)
——, and Root, W. L., "Representations of Vector-Value Random Processes," Lincoln Lab., M.I.T., Lexington, Mass., Group Rept. 55–21; March 7, 1960. Also, *J. Math. Phys.*, vol. 39, pp. 211–216; October, 1960. (Jul 155)
Kharkevich, A. A., "Telecommunications," Pergamon Press, Ltd., London, England; 1957. (Jan 19)
Kielson, J., Mermin, N. D., and Bello, P., "A theorem on cross correlation between noisy channels," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 77–79; June, 1959. (Jul 132)
Kilmer, W. L. "Some Results on best recurrent-type binary error-correcting codes," 1960 IRE INTERNATIONAL CONVENTION RECORD, pt. 4, pp. 135–147. (Jan 8)
Kirsch, R. A., *et al.*, "Experiments in processing pictorial information with a digital computer," *Proc. Eastern Joint Computer Conf.*, December 9–13, 1957, Washington, D. C.; pp. 221–229. (Jul 135)
Knapp, C. F.: See Schreiber, W. F.
Koepcke, R. W.: See Kalman, R. E.
Kohlenberg, A., "Notes on the Zero Distribution of Gaussian Noise," M.I.T. Lincoln Lab., Lexington, Mass., Tech. Memo 44; October, 1953. (Jul 151)
Kolmogoroff, A. A.: See Gnedenko, B. V.
Kolmogorov, A., "Foundations of Probability Theory," Chelsea Publishing Co., New York, N. Y.; 1956. (Oct 233)
Kotel'nikov, V. A., "The Theory of Optimum Noise Immunity," McGraw-Hill Book Co., Inc., New York, N. Y.; 1959. (Jul 157)
Kovasznay, L. S. G., and Arman, A., "Optical autocorrelation measurement of two-dimensional random patterns," *Rev. Sci. Instr.*, vol. 28, pp. 793–797; 1957. (Jul 135)
Kraft, L. G., "A Device for Quantizing, Grouping, and Coding Amplitude Modulated Pulses," M.I.T., Cambridge, Mass., M.S. thesis in elec engrg.; 1949. (Jan 38)
Kramer, H. P.:See Mathews, M. V.
Krein, M. G.: See Gohberg, I. C.
Kretzner, E. R.: See Michel, W. S.
Kullback, S.. "Information theory and Statistics," John Wiley and Sons, Inc., New York, N. Y.; 1959. (Apr 71)
Kurz, L., "An Optimization Procedure for a Single-Link Unidirectional Digital Communication System in the Presence of Additive Gaussian and Impulsive Noise and for Detection Independent of Fading," New York University, New York, N. Y., Contract No. AF 19(604)-6168, 3rd and 4th Scientific Repts; September, 1960. Also, College of Engrg., New York University, Engrg. Sc.D. dissertation; 1961. (Oct 223)

L

Lagrange, J. L., *Nouv. Mem. Ac. Berlin*, pp. 183–272; 1775. (Jan 38)
Lampard, D.: See Barrett, J.
Lampard, D. G., "The proability distribution for the filtered output of a multiplier whose inputs are correlated, stationary, Gaussian time series," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 4–11; March, 1956. (Jan 14)
Landau, E., "Elementary Number Theory," Chelsea Publishing Co., New York, N. Y.; 1958 (Reprint). (Jan 51)
Laning, J., and Battin, R., "Random Processes in Automatic Control," McGraw-Hill Book Co., Inc., New York, N. Y.; 1956. (Oct 267)

Laning, J. H., Jr., "Prediction and Filtering in the Presence of Gaussian Interference," Instrumentation Lab., M.I.T., Cambridge, Mass., Tech. Rept. R-27; October, 1951. (Jul 143) (Oct 257)
Lawson, J. L.: See Uhlenbeck, G. E.
Lawton, J. G., "Theoretical error rates of 'differentially coherent' binary and 'Kineplex' data transmission systems," PROC. IRE, vol. 47, pp. 333–334; February, 1959. (Jul 138)
——: See Cahn, C. R.
Lee, C. Y., "Some properties of nonbinary error-correcting codes," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 77–82; June, 1958. (Jan 51) (Jul 130)
Lehmann, E. L., "Testing Statistical Hypotheses," John Wiley and Sons, Inc., New York, N. Y., p. 65; 1959. (Apr 71)
——, "The power of rank tests," *Ann. Math. Stat.*, vol. 24, pp. 23–43; March, 1953. (Apr 71)
Leipnik, R., "Integral equations, biorthonormal expansions, and noise," *SIAM J.*, vol. 7, pp. 6–30; March, 1959. (Jul 132, 143)
——, "The effect of instantaneous nonlinear devices on cross correlation," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 73–76; June, 1958. (Jul 132)
——, "The extended entropy uncertainty principle," *Inform. and Control*, vol. 3, pp. 18–25; March, 1960. (Jul 130)
Leondes, C. T.: See Hseih, H. C.
Lerner, R. M., "Signals with uniform ambiguity functions," 1958 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 27–36. (Jul 146)
——, "The representation of signals," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 197–216; May, 1959. (Jul 133)
——: See Price, R.
Linden, D. A., and Abramson, N. M., "A generalization of the sampling theorem," *Inform. and Control*, vol. 3, pp. 26–31; March, 1960. (Jul 130)
Littlewood, J. E.: See Hardy, G. H.
Lloyd, S., "Binary block coding," *Bell Sys. Tech. J.*, vol. 36, pp. 517–535; March, 1957. (Jan 4)
Loeve, M., "Probability Theory Foundations, Random Sequences," D. Van Nostrand Co., Inc., Princeton, N. J.; 1955. (Oct 215, 223)
Lyons, D. H.: See Kelly, E. J.

M

MacDonald, D. K. C., "Some statistical properties of random noise," *Proc. Cambridge Phil. Soc.*, vol. 45, pp. 368–372; July, 1949. (Oct 265)
MacDonald, J. E., "Design methods for maximum minimum-distance error-correcting codes," *IBM J. Res. and Dev.*, vol. 4, pp. 43–57; January, 1960. (Oct 223, 276)
MacLane, S.: See Birkhoff, G.
McFadden, J. A., "The axis-crossing intervals of random functions," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 146–150; December, 1956.
——, "The axis-crossing intervals of random functions, II," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 14–24; March, 1958. (Jul 150, 151, 153)
——, "The fourth product moment of infinitely clipped noise," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 159–162; December, 1958. (Jul 153)
——, "The probability density of the output of an RC filter when the input is a binary random process," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 174–178; December, 1959. (Jul 132, 153)
——, "Two expansions for the quadrivariate normal integral," *Biometrika*, vol. 47, pp. 325–333; December, 1960. (Jul 152)
——, "Urn models of correlation and a comparison with the multivariate normal integral," *Ann. Math. Stat.*, vol. 26, pp. 478–489; September, 1955. (Jul 152)
McLachlan, D., "Description mechanics," *Inform. and Control*, vol. 1, pp. 240–266; September, 1958. (Jul 135)
McMillan, B., "Introduction to Formal Realizability Theory," Bell Telephone System, Mono. No. 1994; May, 1952. (Jul 189)
——, "Two inequalities implied by unique decipherability," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 115–116; December, 1956. (Jan 38) (Apr 118)
Magnus, W.: See Oberhettinger, F. Also Tricomi, F. G.
Manasse, R.: See Price, R.
Mandelbrot, B., "On recurrent noise limiting coding," *Symp. on Information Networks*, New York, N. Y., April, 1954; pp. 205–221.
Mann, H. B., "A theory of estimation for the fundamental random process and the Ornstein-Uhlenbeck process," *Sankhya*, vol. 13 pt. 4, pp. 325–358; June, 1954. (Jul 143)
—— and Moranda, P. B., "On the efficiency of the least squares estimates of parameters in the Ornstein-Uhlenbeck process," *Sankhya*, vol. 13, pt. 4, pp. 351–358; June, 1954. (Jul 143)
Marcum, J. I., "A Table of Q-Functions," RAND Corp., Rept. RM-339; January 1, 1950. (Jan 6, 16)
Marcus, R. S., "Discrete Noiseless Coding," M.I.T., Cambridge, Mass., M.S. thesis in elec. engrg.; 1957. (Jan 38)
Marden, M., "The Geometry of the Zeros of a Polynomial in a Complex Variable," *Am. Math. Soc.*, New York, N. Y.; 1949. (Jan 38)
Masani, P.: See Weiner, N.
Masonson, M., "Binary transmission through noise and fading," 1957 IRE NATIONAL CONVENTION RECORD, pt. 2, pp. 69–82. (Jul 138)
Mathews, M. V., and Kramer, H. P., "A linear coding for transmitting a set of correlated signals," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 41–46; September, 1956. (Oct 215)
Mattson, R. L., "A self-organizing binary system," *Proc. Eastern Joint Computer Conf.*, Boston, Mass., December 1–3; 1959, pp. 212–217. (Jul 135)
Max, J., "Mismatching Filters to Improve Resolution in Radar," Group Rept. No. 36-32, M.I.T. Lincoln Lab., Lexington, Mass., October 1, 1958. (Jul 139)
Meggitt, J. E., "Error correcting codes for correcting bursts of errors," *IBM J. Res. and Dev.*, vol. 4; July, 1960. (Jan 53) (Oct 244)
Melas, C. M., "A new group of codes for correction of dependent errors in data transmission," *IBM J. Res. and Dev.*, vol. 4; January, 1960. (Jan 53) (Oct 244)
——, "Cyclic code for double error correction," *IBM J. Res. and Dev.*, vol. 4, pp. 364–366; July, 1960. (Jan 53)
Mermin, N. D.: See Kielson, J.
Merriam, C. W., "A class of optimum control systems," *J. Franklin Inst.*, vol. 267, pp. 267–281; April, 1959. (Jul 143)
Mertz, R., "Model of impulsive noise for data transmission," 1960 IRE INTERNATIONAL CONVENTION RECORD, pt. 5, pp. 247–259. (Oct 223)
Metzner, J. J., "Binary relay communication with decision feedback," 1959 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 112–119. (Jul 139)
Michel, W. S., "Statistical encoding for text and picture communication," *Commun. and Electronics*. vol. 77, pp. 33–36; March, 1958. (Jul 135)
——, Fleckenstein, W. O., and Kretzner, E. R., "A coded facsimile system," 1957 IRE WESCON CONVENTION RECORD, pt. 2, pp. 84–93. (Jul 135)
Middleton, D., "A comparison of random and periodic data sampling for the detection of signals in noise," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 234–247; May, 1959. (Jul 139)
——, "An Introduction to Statistical Communication Theory," Internatl. Series in Pure and Appl. Phys., McGraw-Hill Book Co., Inc., New York, N. Y., 1960. (Jan 15, 42) (Apr 71, 72, 98, 105, 107, 108, 109, 111, 112) (Jul 138, 143, 158) (Oct 233, 246, 247, 271)
——, "A note on the estimation of signal waveform," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 86–89; June, 1959 (Jul 132, 143) (Oct 215)
——, "On new classes of matched filters and generalizations of the matched filter concept," IRE TRANS. ON INFORMATION THEORY, vol. IT-6, pp. 349–360; June, 1960. (Jul 139)
——, "On singular and nonsingular (Bayes) tests for the detection of normal stochastic signals in normal noise," IRE TRANS. ON INFORMATION THEORY, vol. IT-7, pp. 105–113; April, 1961. (Oct 233)
——, "On the detection of stochastic signals in additive normal noise—Part I," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 86–122; June, 1957. (Apr 105) (Jul 139, 158) (Oct 215, 233)
——, "On the detection of stochastic signals in additive normal noise—Part II," IRE TRANS. ON INFORMATION THEORY, vol. IT-6, pp. 349–360; June, 1960. (Oct 233)
——, "Statistical criteria for the detection of pulsed carriers in noise: I, II," *J. Appl. Phys.*, vol. 24, pp. 371–378, 379–391; April, 1953. (Apr 71)
——, "Theory of random noise; phenomenological models," *J. Appl. Phys.*, vol. 22, pp. 1153–1163; September, 1951. (Apr 62)
Middleton, D.: See Van Meter, D. Also Van Uleck, J. H.
——, and Bussgang, J. J., "Optimum sequential detection of signals

in noise," IRE TRANS. ON INFORMATION THEORY, vol. IT-1, pp. 5–18; December, 1955. (Apr 105)

and Johnson, V., "A Tabulation of Selected Confluent Hypergeometric Functions," Cruft Lab., Harvard University, Cambridge, Mass., Tech. Rept. No. 140; January 5, 1952. (Apr 66)

Miller, K. S. and Bernstein, R. I., "An analysis of coherent integration and its application to signal detection," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 237–248; December, 1957. (Jul 139)

and Zadeh, L. A., "Solution of an integral equation occurring in the theories of prediction and detection," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 72–75; June, 1956. (Apr 120) (Jul 143)

Minsky, M. L., "Heuristic Aspects of the Artificial Intelligence Problem," Lincoln Lab., M.I.T., Lexington, Mass.; 1956. (Apr 102)

Mood, A. M., "Introduction to the Theory of Statistics," McGraw-Hill Book Co., Inc., New York, N. Y.; 1950. (Jul 195)

Moore, E. F.: See Gilbert, E. N.

Moranda, P. B.: See Mann, H. B.

Morgan, K.: See Chang, S. S. L.

Moyal, J. E., "Stochastic processes and statistical physics," *J. Roy. Stat. Soc., Ser. B.*, vol. 11, pp. 150–210; 1949. (Apr 72)

Mudgett, W. L.: See Bussgang, J. J.

Murray, A. E., "A review of the perceptron program," *Proc. Natl. Electronics Conf.*, vol. 15, pp. 346–356; October, 1959. (Jul 135)

N

Nagy, B.: See Riesz, F.

Neisser, V., and Weene, P., "A note on human recognition of hand-printed characters," *Inform. and Control*, vol. 3, pp. 191–196; June, 1960. (Jul 135)

Nesbeda, P.: See Bussgang, J. J.

Nilsson, N. J., "An application of the theory of games to radar reception problems," 1959 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 130–140. (Jul 139)

Nuttall, A. H., "Theory and Application of the Separable Class of Random Processes," Research Lab. of Electronics, M.I.T., Cambridge, Mass., Rept. No. 343; May, 1958. (Jul 143)

O

Oberhettinger, F., and Magnus, W., "Special Functions of Mathematical Physics," Chelsea Publishing Co., New York, N. Y., p. 118; 1949. (Jan 16)

Oberhettinger, F.: See Tricomi, F. G.

Ono, Y., and Yasuura, K., "Synthesis of finite passive $2n$-terminal networks with prescribed scattering matrices," *Mem. Kyushu Univ. (Engineering), Japan*, vol. 14, no. 2, pp. 125–177; 1954. (Jul 189)

P

Palmer, D. S., "Properties of random functions," *Proc. Cambridge Phil. Soc.*, vol. 52, pp. 672–686; October, 1956. (Jul 153)

Park, J. H., Jr., "Statistical Estimation of Normalized Linear Signal Parameters," Rad. Lab., Johns Hopkins University, Baltimore, Md., Tech. Rept. AF-72; December, 1959 (Unclassified). (Apr 98)

——: See Glaser, E. M.

Parzen, E., "Statistical Inference on Time Series by Hilbert Space Methods, I," Appl. Math. and Stat. Lab., Stanford, Calif., Tech. Rept. 23; January, 1959. (Apr 119)

Patterson, G. W.: See Sardinas, A. A.

Peach, L. C.: See Cohn, G. I.

Perry, K. E., "An error-correcting encoder and decoder for phone line data," 1958 IRE WESCON CONVENTION RECORD, pt. 4, pp. 21–26. (Jul 130)

Peterson, W. W., "Encoding and error-correction procedures for the Bose-Chaudhuri codes," IRE TRANS. ON INFORMATION THEORY, vol. IT-6, pp. 459–470; September, 1960. (Jul 130)

——, "Error-Detecting and Error-Correcting Codes," Technology Press, Cambridge, Mass., Research Mono.; 1960 (Jul 131)

——, "Error-Detecting and Correcting Codes," Technology Press, Cambridge, Mass., and John Wiley and Sons, Inc., New York, N. Y.; 1961. (Oct 276)

——, Birdsall, T. G., and Fox, W. C., "Theory of signal detectability," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 171–212; September, 1954. (Apr 71) (Oct 223)

—— and Brown, D. T., "Cyclic codes for error detection," PROC. IRE, vol. 49, pp. 228–235; January, 1961. (Oct 244)

—— and ——, "Encoding and error-correction procedures for the Bose-Chaudhuri codes," IRE TRANS. ON INFORMATION THEORY, vol. IT-6, pp. 459–470; September, 1960. (Oct 244)

Peterson, S. W., and Fontaine, A. B., "Group code equivalence and optimum codes," IRE TRANS. ON INFORMATION THEORY, vol. IT-5, pp. 60–70; May, 1959. (Jan 8, 9, 11) (Jul 130)

Pierce, J. N., "A Markoff envelope process," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 163–166; December, 1958. (Jul 132)

Pierce, J. N., "Theoretical diversity improvement in frequency-shift keying," PROC. IRE, vol. 46, pp. 903–910; May, 1958. (Jul 138, 164)

Pierce, J. R., and Karlin, J. E., "Reading rates and the information rate of a human channel," *Bell Sys. Tech. J.*, vol. 36, pp. 497–516; March, 1957. (Jul 135)

Pitman, E. J. G., "Lecture Notes on Nonparametric Statistical Interference," Columbia University, New York, N. Y.; Spring, 1948. (Apr 71)

Pollak, H. O.: See Grenander, U.

Polya, G.: See Hardy, G. H.

Portmann, P.: See Scherer, E.

Powers, K. H., and Staras, H., "Some relations between television picture redundancy and bandwidth requirements," *Commun. and Electronics*, vol. 76, pp. 492–496; September, 1957. (Jul 135)

Prange, E., "Coset Equivalence in the Analysis and Decoding of Group Codes," AF Cambridge Research Center, Cambridge, Mass., Tech. Note AFCRC-TR-59-164; June, 1959. (Jul 131)

Preston, G. W., "The Advanced Theory of Radar Measurements," General Atronics Corp., Final Rept. No. 799-207-12 to the Rome Air Dev. Center on Contract AF 30(602)-2120; August 20, 1960. (Oct 246)

——, "The equivalence of optimum transducers and sufficient and most efficient statistics," *J. Appl. Phys.*, vol. 24, pp. 841–844; July, 1953. (Apr 118)

Price, R., "A note on the envelope and phase-modulated components of narrow-band Gaussian noise," IRE TRANS. ON INFORMATION THEORY, vol. IT-1, pp. 9–15; September, 1955. (Jan 42)

——, "Error Probabilities for the Ideal Detection of Signals Perturbed by Scatter and Noise," Lincoln Lab., M.I.T., Lexington, Mass., Group Rept. 34–40; October 3, 1955. (Oct 233)

——, "A useful theorem for nonlinear devices having Gaussian inputs," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 69–72; June, 1958. (Jul 132)

——, "Notes on Ideal Receivers for Scatter Multipath," Lincoln Lab., M.I.T., Lexington, Mass., Group Rept. 34–39; May 12, 1955. (Oct 233)

——, "Optimum detection of random signals in noise with applications to scatter-multipath communication. Part I," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 125–135; December, 1956. (Jul 161) (Oct 233)

——, "Radiometer Techniques in Radar," Lincoln Lab., M.I.T., Lexington, Mass., Rept. No. 34-G-0003; June 10, 1960. (Jul 139)

——, "The Venus Radar Experiment," printed paper presented at the 9th General Assembly of A. G. A. R. D., N. A. T. O., Aachen, Germany; September 21, 1959. (Jul 139)

—— and Green, P. E., "A communication technique for multipath channels," PROC. IRE, vol. 46, pp. 555–569; March, 1958. (Jan 19, 41) (Apr 98)

—— and ——, "Signal Processing in Radar Astronomy—Communication via Fluctuating Multipath Media," Lincoln Lab., M.I.T., Lexington, Mass., Rept. No. 234; 1960. (Oct 233)

——, Manasse, R., and Lerner, R. M., "Loss of signal detectability in band-pass limiters," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 34–38; March, 1958. (Jan 39) (Jul 139)

——, *et al.*, "Radar echoes from Venus," *Science*, vol. 129, pp. 751–753; March 20, 1959. (Jul 139)

Pugachev, V. S., "Method of solving the basic integral equation of the statistical theory of optimum systems in closed form," *Prik. Mat. i Mah.*, vol. 23, pp. 3–14; January, 1959. (In Russian.) (Apr 120)

R

Raemer, H. R., and Blyth, R., "Mathematical Model of Sonar Target Returns in Reverberation and Water Noise," Cook Research Labs., Morton Grove, Ill., Phase 1, Final Rept., pt. 3, Contract NObsr 77614, U. S. Navy Bureau of Ships; March 29, 1960. (Oct 265)

Ragazzini, J. R.: See Zadeh, B. Also Zadeh, L. A.

Ragonese, F.: See Flores, I.

Ray-Chaudhuri, D. K., and Bose, R. C., "Further results on error-correcting binary group codes," *Inform. and Control*, vol. 3, pp. 279–290; September, 1960. (Jul 130)
—— and ——, "On a class of error correcting binary group codes," *Inform. and Control*, vol. 3, pp. 68–79; March, 1960. (Jul 130) (Oct 244, 276)
Reed, I. S., "An Approximation the Output Signal-to-Noise Ratio of a Signal, Passed through a Band-Pass Limiter, Followed by a Narrow-Band Filter," Lincoln Lab., M.I.T., Lexington, Mass., Group Rept. 47.17; June 2, 1958. (Jan 40)
——, "A class of multiple-error-correcting codes," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 38–49; September, 1954. (Jul 131)
——, Kelly, E. J., and Root, W. L., "The Detection of Radar Echoes in Noise. Part I: Statistical Preliminaries and Detector Design. Part II: The Accuracy of Radar Measurements," Lincoln Lab., M.I.T., Lexington, Mass., Tech. Rept. No. 158, June 20, 1957; Tech. Rept. No. 159, July 19, 1957. (Jul 139)
Reich, E., and Swerling, P., "Detection of a sine wave in Gaussian noise," *J. Appl. Phys.*, vol. 24, pp. 289–296; March, 1953. (Apr 71)
Reiger, S., "Error rates in data transmission," PROC. IRE, vol. 46, p. 919; May, 1958. (Jan 5) (Jul 138)
Rice, S. O., "Distribution of the duration of fades in radio transmission," *Bell Sys. Tech. J.*, vol. 37, pp. 581–635 (114); May, 1958. (Jul 150)
——, "Mathematical analysis of random noise," *Bell Sys. Tech. J.*, vol. 23, pp. 282–332, July, 1944; vol. 24, pp. 46–156, January, 1945. (Apr 73) (Jul 150, 151, 192)
——, "Mathematical analysis of random noise," in "Noise and Stochastic Processes," N. Wax, Ed., Dover Publications, Inc., New York, N. Y.; 1954. (Jan 21) (Apr 71)
——, "Statistical properties of a sine wave plus random noise," *Bell Sys. Tech. J.*, vol. 27, pp. 109–157; January, 1948. (Apr 71,79)
Riesz, F., and Nagy, B., "Functional Analysis," Frederick Ungar Publishing Co., New York, N. Y.; 1955. (Oct 233)
Robbins, H. M., "An extension of Wiener filter theory to partly sampled systems," IRE TRANS. ON CIRCUIT THEORY, vol. CT-6, pp. 362–370; December, 1959. (Jul 144)
Root, W. L., and Davenport, W. B., "Introduction to the Theory of Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., p. 288; 1958. (Jan 39, 48) (Jul 138, 143, 155, 192) (Oct 222, 233, 260, 265)
—— and Davenport, W. B., Jr., "Random Signals and Noise," McGraw-Hill Book Co., Inc., New York, N. Y., p. 166, Eq. (8-114); 1958. (Apr 66, 74)
——: See Kelly, E. J. Also Reed, I. S.
Rosenblatt, F., "Perception simulation experiments," PROC. IRE vol. 48, pp. 301–309; March, 1960. (Jul 135)
——: See Grenander, U.
Rothstein, J., "Analysis of binary time series in periodic functions," IRE TRANS. ON ELECTRONIC COMPUTERS, vol. EC-8, p. 229; June, 1959. (Jan 12)

S

Sacks, G. E., "Multiple error correction by means of parity checks," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 145–147 December, 1958. (Jan 10)
Safran, H.: See Bussgang, J. J.
Samuel, P.: See Zariski, O.
San Soucie, R. L.: See Green, J. H.
Sardinas, A. A., and Patterson, G. W., "A necessary and sufficient condition for the unique decomposition of coded messages," 1953 IRE CONVENTION RECORD, pt. 8, pp. 104–108. (Jan 38)
Scherer, E., Filipowsky, R., and Portmann, P., "Improvements Obtained by a Quaternary Code and Decision System," presented at the 3rd AeroCom. Symp., Rome-Utica, N. Y.; November 8, 1957. (Jan 3)
Schreiber, W. F., "Probability Distributions of Television Signals," Harvard University, Cambridge, Mass., Ph.D. dissertation; 1952. (Apr 99)
——, "The measurement of third-order probability distributions of television signals," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 94–105; September, 1956. (Apr 99)
—— and Knapp, C. F., "TV bandwidth reduction by digital coding," 1958 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 88–99. (Jul 131, 135)
Schutzenberger, M. P., "Contribution aux Applications Statistiques de la Theorie de L'information," Universite de Paris, Paris, France, vol. 3, Fascicules 1–2; 1954.
Schwartz, L., "Theorie des Distributions," Herman and Co., Paris, France, vols. 1, 2; 1950. (Oct 223)
Schwartz, L. S.: See Chang, S. S. L.
Schwartz, L. S., Harris, B., and Hauptschein, A., "Information rate from the viewpoint of inductive probability," 1959 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 102–111. (Jul 139)
——, *et al.*, "Optimum decision feedback circuits," 1957 IRE NATIONAL CONVENTION RECORD, pt. 2, pp. 3–10. (Jul 139)
Selfridge, O. G., "Pandemonium: a Paradigm for Learning," Lincoln Lab., M.I.T., Lexington, Mass., JA 1140; 1958. (Apr 102)
——, "Pattern Recognition and Learning," presented at Symp. on Information Theory, London, England; 1955. (Apr 102)
——, "Pattern recognition and modern computers," *Proc. Western Joint Computer Conf.*, Los Angeles, Calif., March 1–3, 1955; pp. 82–84. (Apr 102)
Shannon, C. E., "A mathematical theory of communication," *Bell Sys. Tech. J.*, vol. 27, pp. 379–423, July 1948; pp. 623–656, October, 1948. (Jan 38) (Oct 223)
——, "Channels with side-information at the transmitter," *IBM J.*, vol. 2, pp. 289–293; October, 1958. (Jul 131)
——, "Certain results in coding theory for noisy channels," *Inform. and Control*, vol. 1, pp. 6–25; September, 1957. (Jul 131)
——, "Probability of error for optimal codes in a Gaussian channel," *Bell Sys. Tech. J.*, vol. 38, pp. 611–656; May, 1959. (Jul 131)
——, "The Mathematical Theory of Communication," University of Illinois Press, Urbana; 1949. (Jul 131)
Shapiro, H. S.: See Slotnick, D. L.
Sherman, S., "Non-mean-square error criteria," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 125–126; September, 1958. (Jul 144)
Shinbrot, M., "A generalization of a method for the solution of the integral equation arising in optimization of time-varying linear systems with nonstationary inputs," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 220–224; December, 1957. (Jul 143)
Shuey, R. L.: See White, G. M.
Siebert, W. M., "A radar detection philosophy," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 204–221; September, 1956 (Jul 146)
——, "Studies of Woodward Uncertainty Function," Research Lab. of Electronics, M.I.T., Cambridge, Mass., Quarterly Prog. Rept.; April 15, 1958. (Jul 146)
——, "Some applications of detection theory to radar," 1958 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 5–14. (Jul 139)
Siegert, A. J. F., "A systematic approach to a class of problems in the theory of noise and other random phenomena—Part II, Examples," IRE TRANS. ON INFORMATION THEORY, vol. IT-3, pp. 38–43; March, 1957. (Jul 132)
——, "A systematic approach to a class of problems in the theory of noise and other random phenomena—Part III, Examples," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 4–14; March, 1958. (Jul 132)
——, "On the Roots of Markoffian Random Functions," RAND Corp., Santa Monica, Calif., Rept. No. RM-447; September, 1950. (Jul 150)
——: See Darling, D. A.
Silverman, R. A., "On binary channels and their cascades," IRE TRANS. ON INFORMATION THEORY, vol. IT-1, pp. 19–27; December, 1955. (Oct 223)
——: See Kay, I.
Singer, J. R., "Information theory and the human visual system," *J. Opt. Soc. Am.*, vol. 49, pp. 639–640; June, 1959. (Jul 135)
Slepian, D., "A class of binary signaling alphabets," *Bell Sys. Tech. J.*, vol. 35, pp. 203–234; January, 1956. (Jan 3, 9) (Apr 114) (Jul 131, 171) (Oct 223)
——, "A note on two binary signaling alphabets," IRE TRANS. ON INFORMATION THEORY, vol. IT-2, pp. 84–86; June, 1956. (Jul 131)
——, "Some comments on the detection of Gaussian signals in Gaussian noise," IRE TRANS. ON INFORMATION THEORY, vol. IT-4, pp. 65–68; June, 1958. (Apr 105) (Jul 132, 139)
——: See Grenander, U. Also Kac, M.
Slotnick, D. L., and Shapiro, H. S., "On the mathematical theory of error-correcting codes," *IBM J. Res. and Dev.*, vol. 3, pp. 25–34; January, 1959. (Jan 2, 51) (Apr 114)
Sondhi, M. M: See Benedict, T. R.
Speiser, A., "Die Theorie der Gruppen von endlicher Ordnung," Dover Publications, Inc., New York, N. Y.; 1945 (Jan 2)
Spilker, J. J., Jr., "Theoretical bounds on the performance of sampled data communications systems," IRE TRANS. ON CIRCUIT THEORY, vol. CT-7, pp. 335–341; September, 1960. (Oct 269)

Stam, A. J., "Some inequalities satisfied by the quantities of information of Fisher and Shannon," *Inform. and Control*, vol. 2, pp. 101–112; June, 1959. (Jul 131)
Staras, H.: See Powers, K. H.
Stearns, S. D., "A method for the design of pattern recognition logic," IRE Trans. on Electronic Computers, vol. EC-9 pp. 48–53; March, 1960. (Jul 135)
Stern, T. E., and Friedland, B., "Application of modular sequential circuits to single error-correcting p-nary codes," IRE Trans. on Information Theory, vol. IT-5, pp. 114–123; September, 1959 (Jan 53)
Stewart, J. L., and Westerfield, E. C., "A theory of active sonar detection," Proc. IRE, vol. 47, pp. 872–881; May, 1959. (Jul 139)
Stewart, R. M., "Statistical design and evaluation of filters for the restoration of sampled data," Proc. IRE, vol. 44, pp. 253–257; February, 1956. (Oct 269)
Stone, W. M., Brock, R. L., and Hammerle, K. J., "On the first probability of detection by a radar receiver system," IRE Trans. on Information Theory, vol. IT-5, pp. 9–11; March, 1959. (Jul 138)
Storer, J., and Galejs, J., "Effects of Limiting on the False Alarm Rate of an Envelope Detector," Appl. Research Lab., Sylvania Electric Products, Inc., Waltham, Mass., Appl. Research Memo. 135; June, 1958. (Apr 64)
Strum, P. D., "Considerations in high-sensitivity microwave radiometry," Proc. IRE, vol. 46, pp. 43–53; January, 1958. (Jul 139)
Stumpers, F. L. M., "Distortion of frequency modulated signals in electrical networks," *Commun. News*, vol. 9, pp. 82–92; April, 1948. (Jan 26)
Sunde, E. D., "Ideal binary pulse transmission by AM and FM," *Bell Sys. Tech. J.*, vol. 38, pp. 1357–1426; November, 1959. (Jul 138)
Swerling, P., "Detection of fluctuating pulsed signals in the presence of noise," IRE Trans. on Information Theory, vol. IT-3, pp. 175–178; September, 1957. (Jul 139)
——, "Optimum linear estimation for random processes as the limit of estimates based on sampled data," 1958 IRE WESCON Convention Record, pt. 4, pp. 158–163. (Jul 144)
——, "Paradoxes related to the rate of transmission of information," *Inform. and Control.*, vol. 3, p. 351; December, 1960. (Apr 105)
——, "Parameter estimations for waveforms in additive Gaussian noise," *SIAM J.*, vol. 7, pp. 152–166; June, 1959. (Jul 132)
——, "The resolvability of point sources," *Proc. Symp. on Decision Theory and Applications to Electronic Equipment Dev.*, Rome Air Dev. Center, Griffiss AFB, Rome, N. Y., vol. 1, RACD-TR-60-70A; April, 1960. (Oct 245)
——: See Reich, E.
Szego, G., "Bertrage Zur Theorie der Toeplitzschen Formen," *Math. Z.*; 1920. (Oct 233)

T

Tersoff, A. I., "Automatic registration in high-speed character-sensing equipment," *Proc. Eastern Joint Computer Conf.*, Washington, D. C., December 9–13, 1957; pp. 232–237. (Jul 135)
Thomas, J. B., and Wolf, J. K., "On the statistical detection problem for multiple signals," IRE Trans. on Information Theory, to be published. (Jul 155)
—— and Wong, E., "On the statistical theory of optimum demodulation," IRE Trans. on Information Theory, vol. IT-6, pp. 420–425; September, 1960. (Apr 118) (Jul 155)
—— and Zadeh, L. A., "Note on an integral equation occurring in the prediction, detection, and analysis of multiple time series," IRE Trans. on Information Theory, vol. IT-7, pp. 118–120; April, 1961. (Jul 155)
Thomasian, A. J.: See Blackwell, D.
Tompkins, D. N., "Codes with Zero Correlation," Engrg. Div., Hughes Aircraft Co., Culver City, Calif., Tech. Memo 651; June, 1960. (Oct 255)
Tricomi, F. G., Erdelyi, A., Magnus, W., and Oberhettinger, F., "Table of Integral Transforms," McGraw-Hill Book Co., Inc.; 1954. Also, "Higher Transcendental Functions," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953. (Jan 16)
Tsvetnov, V. V., "Statistical properties of signals and noise in two channel phase systems," *Radiotekh. i Elektron.*, vol. 12, no. 5, pp. 12–30; 1957. (In Russian.) (Oct 267)
Tukey, J. W.: See Balckman, R. B.
Tung, F., "A Class of Optimum Nonlinear Filters for Quantized Inputs," Dept. of Elec. Engrg., Columbia University, New York, N. Y., Tech. Rept. T-2/N; July, 1960. (Oct 260)
Turin, G. L., "An introduction to matched filters," IRE Trans. on Information Theory, vol. IT-6, pp. 311–329; June, 1960. (Jan 42, 52) (Apr 88) (Oct 254)
——, "Communication through noisy, random-multipath channels," 1956 IRE Convention Record, pt. 4, pp. 154–156. (Jul 154, 163) (Oct 233)
——, "Error probabilities for binary symmetric ideal reception through nonselective slow fading and noise," Proc. IRE, vol. 46, pp. 1603-1619; September, 1958. (Jul 138, 164) (Oct 233)
——, "On the estimation in the presence of noise of the impulse response of a random, linear filter," IRE Trans. on Information Theory, vol. IT-3, pp. 5–10; March, 1957. (Oct 252)
——, "Some computations of error rates for selectively fading multipath channels," *Proc. Natl. Electronics Conf.*, vol. 15, (in press). (Jul 138)

U

Uhlenbeck, G. E., and Lawson, J. L., "Threshold Signals," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y.; 1950. (Jan 42) (Apr 66, 72) (Jul 152, 192)
Uhr, L., "Machine Perception of Printed and Handwritten Forms by Means of Procedures for Assessing and Recognizing Gestalts," presented at 14th Natl. Meeting of Assoc. for Comp. Mach., Cambridge, Mass.; 1959. (Apr 102)
Ule, L. A., "A theory of weighted smoothing," IRE Trans. on Information Theory, vol. IT-3, pp. 131–135; June, 1957. (Jul 144)
Ulrich, W., "Non-binary error correction codes," *Bell Sys. Tech. J.*, vol. 36, pp. 1341–1388; November, 1957. (Jan 2) (Jul 131)
Unger, S. H., "Pattern detection and recognition," Proc. IRE, vol. 47, pp. 1737–1752; October, 1959. (Apr 102) (Jul 135)
Uttley, A. M., "Imitation of pattern recognition and trial-and-error learning in a conditional probability computer," *Rev. Mod. Phys.*, vol. 31, pp. 546–548; April, 1959. (Jul 135)

V

Van der Pol, B., "The fundamental principles of frequency modulation," *J. IEE*, vol. 93, pt. 3, pp. 153–158; May, 1946. (Jan 26)
Van der Waerden, B. L., "Modern Algebra," Frederick Ungar Publishing Co., New York, N. Y.. vol. 1, pp. 111–115; 1953. (Oct 256)
Van Meter, D., and Middleton, D., "Detection and extraction of signals in noise from the point of view of statistical decision theory," *J. Soc. Ind. Appl. Math.*, vol. 3, pp. 192–253, December, 1955; vol. 4, pp. 86–119, June, 1956. (Apr 98) (Oct 223)
—— and ——, "Modern statistical approaches to reception in communication theory," IRE Trans. on Information Theory, no. IT-4, pp. 119–145; September, 1954. (Jan 39) (Oct 215)
Van Vleck, J. H., and Middleton, D., "A theoretical comparison of visual, aural, and meter reception of pulsed signals in the presence of noise," *J. Appl. Phys.*, vol. 17, pp. 940–971; November, 1946. (Jul 192)
Varchaver, B. A., "On the theory of transmitting signals with multiple discrete values," *Radiotekhnika (Moscow)*; January, 1959. (Jul 192)

W

Wald, A., "Statistical Decision Functions," John Wiley and Sons, Inc., New York, N. Y.; 1950. (Oct 215)
Waters, W. M., "Pulse Separation by Amplitude Filters," Rad. Lab., Johns Hopkins University, Baltimore, Md., Tech. Rept. No. AF-77; May, 1960. (Jul 192)
Wax, N., "On upper bounds for error detecting and error correcting codes of finite length," IRE Trans. on Information Theory, vol. IT-5, pp. 168–174; December, 1959. (Oct 270, 276)
Weene, P.: See Neisser, V.
Weiner, N., "Mathematical Problems of Communication Theory," Summer Session Lecture Notes, M.I.T., Cambridge, Mass.; 1953. (Jul 143)
——, "Nonlinear Problems in Random Theory," John Wiley and Sons, Inc., New York, N. Y., 1958. (Jul 132, 143)
—— and Masani, P., "The prediction theory of multivariate stochastic processes," *Acta Math.*, vol. 98, 1957; vol. 99, 1958. (Jul 133, 189)
Wernikoff, R. E., "A Theory of Signals," Research Lab. of Electronics, M.I.T., Cambridge, Mass., Rept. No. 331; January, 1958. (Jul 144)

Y

Z

A STATEMENT OF EDITORIAL POLICY

The IRE TRANSACTIONS ON INFORMATION THEORY is a quarterly journal devoted to the publication of papers on the transmission, processing, and utilization of information. The exact subject matter of acceptable papers is intentionally, by editorial policy, not sharply delimited. Rather, it is hoped that as the focus of research activity changes, a flexible policy will permit the TRANSACTIONS to follow suit and that it will continue to serve its readers with timely articles on the fundamental nature of the communication process. Topics of current appropriateness include the coding and decoding of digital and analog communication transmissions, studies of random interferences and of information bearing signals, analyses and design of communication and detection systems, pattern recognition, learning, automata, and other forms of information processing systems.

Papers can be of two kinds, tutorial or research, and should be so indicated. The former must be well-written expositions summarizing the state of a field in which research is still in progress, or else unify results scattered in the literature. Research papers must be original contributions not published elsewhere. They must present new methods, concepts, or ideas, or extend old ones to new areas of applicability; or, they must present new data, findings or inventions, or solve new problems of more than casual interest. They will not be accepted if, in the view of the reviewers and editors, they constitute a straightforward and easy application of existing theory to a special case of limited interest. It is not necessary that the length of each research paper be great; on the contrary, the submission of short but formal research notes is to be encouraged.

In addition to papers, readers are invited to submit notes to the Correspondence section. These may include such things as early summaries of important work to be published later at greater length, or remarks on material that has already appeared. Contributions in the form of "problem statements" are also sought for the Correspondence section. This category includes problems to which the author knows no solution but suspects that another reader might, conjectures for which a proof or disproof is desired, and so forth.

INFORMATION FOR AUTHORS

Authors are requested to submit editorial correspondence or technical manuscripts to the Editor for possible publication in the PGIT TRANSACTIONS. Papers submitted should include a statement as to whether the material has been copyrighted, previously published, or submitted for publication elsewhere.

The Professional Group wishes to emphasize that publication of a paper in the IRE INTERNATIONAL CONVENTION RECORD precludes its consideration for the TRANSACTIONS.

To expedite reviewing procedures, it is requested that authors submit the original and two legible copies of all written and illustrative material. The manuscript should be double-spaced, and the illustrations drawn in India ink on drawing paper or drafting cloth. Each paper should include a carefully written abstract of not more than 200 words. Papers should be prepared for publication in a matter similar to those intended for the PROCEEDINGS OF THE IRE. Further instructions may be obtained from the Editor. The original copy and drawings of material not accepted for publication will be returned.

All technical manuscripts and editorial correspondence should be addressed to Arthur Kohlenberg, Melpar, Inc., 11 Galen Street, Watertown 72, Mass.

Local Chapter activities and announcements, as well as other nontechnical news items, should be addressed to the PGIT Newsletter, c/o Prof. N. M. Abramson, Electrical Engineering Department, Stanford University, Stanford, Calif.

INSTITUTIONAL LISTINGS

The IRE Professional Group on Information Theory is grateful for the assistance given by the firms listed below and invites application for Institutional Listing from other firms interested in the field of Information Theory.

AIRBORNE INSTRUMENTS LABORATORY, Div. of Cutler-Hammer, Inc., Deer Park, L. I., N. Y.
Research, Development and Production of Electronic Systems and Components

IBM CORPORATION, 590 Madison Ave., New York 22, N. Y.
Information Handling Systems and Applications

IBM RESEARCH, INTERNATIONAL BUSINESS MACHINES CORP., Yorktown Heights, N. Y.
Error Correcting & Detecting Codes, Theory of Assemblies & Automata, Information Networks, Reliability

REPUBLIC AVIATION CORP., Farmingdale, N. Y.
Advanced Aircraft, Space Systems, Missile Systems and Electronics

The charge for an Institutional Listing is $50 per issue or $150 for four consecutive issues. Applications for Institutional Listings and checks (made payable to the Institute of Radio Engineers) should be sent to L. G. Cumming, Institute of Radio Engineers, 1 East 79 Street, New York 21, N. Y.